Classification of Generalized Cusps

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Outline

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   - Description/geometry of cusps
   - Focus on properties to generalize

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2. Properly Convex Manifolds
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   - How do they similar/different to hyperbolic manifolds
3. Generalized Cusps
   - Description/geometry
   - How to classify

Cusps of hyperbolic orbifolds

Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ be a lattice and $M = \mathbb{H}^n/\Gamma$ be a complete hyperbolic n-orbifold.

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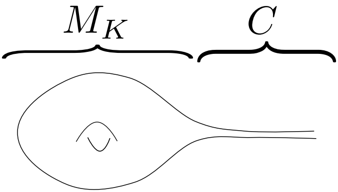
Using the “thik-thin” decomposition M can be decomposed into

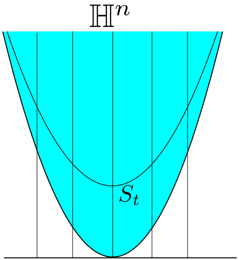
$$M = M_k \bigsqcup_i C_i,$$

where C_i is finitely covered by $T^{n-1} \times [0, \infty)$.



Cusps of hyperbolic manifolds

- Let $\mathbb{H}^n = \{(z, v) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid z > \frac{1}{2}|v|^2\} \subset \mathbb{RP}^n$



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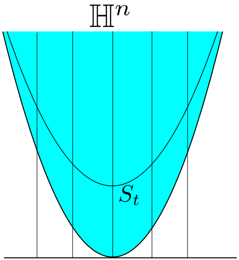
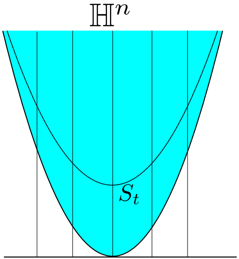
- $\mathbb{H}^n$ is foliated by horospheres
  $S_t = \{(z, v) \in \mathbb{H}^n \mid x = \frac{1}{2}|v|^2 + t\}$, $t > 0$



Consider the following subgroups of $\mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$

$$T = \left\{ \begin{pmatrix} 1 & u & \frac{1}{2}|u|^2 \\ 0 & I & u \\ 0 & 0 & 1 \end{pmatrix} \mid u \in \mathbb{R}^{n-1} \right\}, O = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid A \in O(n-1) \right\}$$

- T acts simply transitively on each S_t



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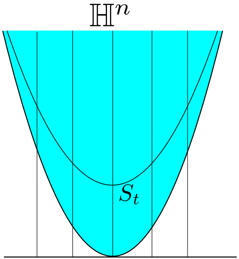

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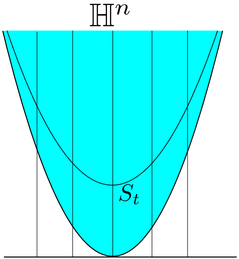
- $G = T \rtimes O$ preserves the foliation leafwise



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Geometry of cusps

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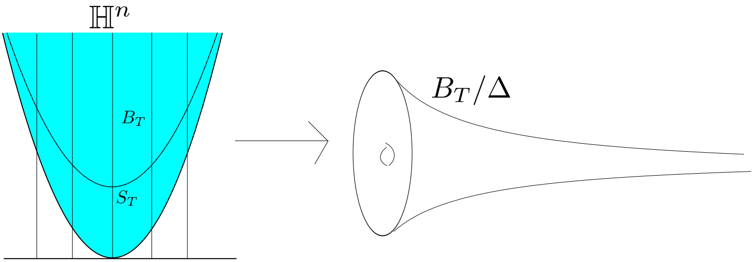

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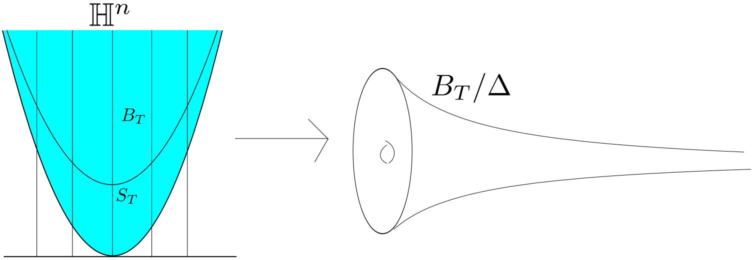
The S_t/Δ give a foliation of C by Euclidean $(n-1)$-orbifolds.



Properly convex manifolds

A subset $\Omega \subset \mathbb{RP}^n$ with non-empty interior is *properly convex* if

1. Ω is convex in $\mathbb{RP}^n$ (intersections with projective lines are connected)
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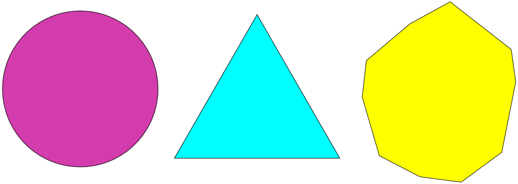
Ω can be realized as a compact, convex subset of $\mathbb{R}^n \subset \mathbb{RP}^n$.



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In general, properly convex domains can have “flats” in their boundary.

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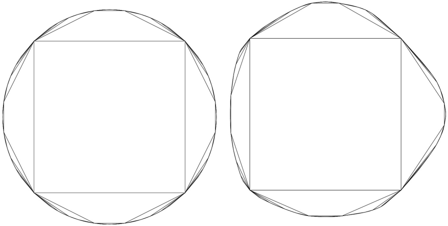

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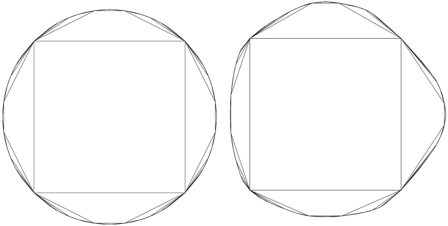

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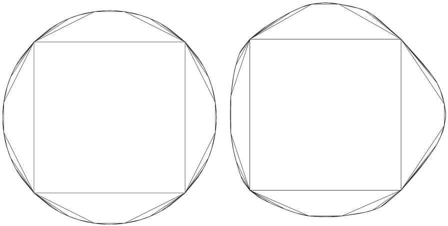

If M has cusps, what does the geometry of the cusps of Ω_t/Γ_t look like if $t \neq 0$? *They are generalized cusps.*

Generalized cusps

A generalized cusp is a properly convex manifold $C = \Omega/\Gamma$ where

- C is diffeomorphic to $\partial C \times [0, \infty)$, with ∂C compact
- $\Gamma \cong \pi_1 \partial C$ is virtually abelian
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Cusps of finite volume hyperbolic manifolds are generalized cusps

Geometry of generalized cusps

Let $W_n = \{(\lambda_1, \ldots, \lambda_n) \mid 0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_n\}$

Given an n-dimensional generalized cusp $C \cong \Omega/\Gamma$ we get

Geometry of generalized cusps

Overview

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Given an n-dimensional generalized cusp $C \cong \Omega/\Gamma$ we get

- $\lambda \in W_n$, unique up to scaling.
- A Lie group $\mathrm{PGL}_{n+1}(\mathbb{R}) \supset G_\lambda \cong \underbrace{T_\lambda}_{\text{translations}} \rtimes \underbrace{O_\lambda}_{\text{point stabilizer}}$ that contains a conjugate of Γ as a lattice.

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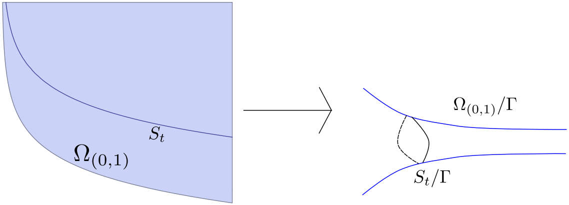
- A G_λ-invariant properly convex domain $\Omega_\lambda \subset \Omega$ (e.g. $B_T \subset \mathbb{H}^n$)
- A foliation of Ω_λ by strictly convex hypersurfaces (horospheres)

A quasi-hyperbolic cusp

- Let $\Omega_{(0,1)} = \{(z,y) \in \mathbb{R} \times \mathbb{R}_+ \mid z > -\log(y)\}$
- $\Omega_{(0,1)}$ is foliated by $S_t = \{(z,y) \in \Omega \mid z = -\log(y) + t\}$ (horospheres)



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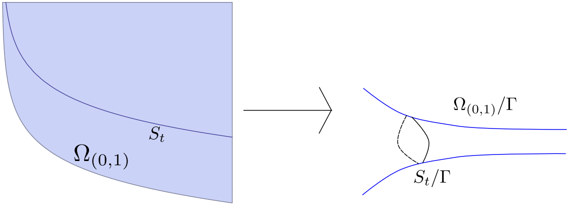
Let Γ be a lattice in the Lie group

$$G_{(0,1)} = \left\{ \begin{pmatrix} 1 & 0 & -u \\ 0 & e^u & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid u \in \mathbb{R} \right\}$$



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- Let $f_\lambda : \mathbb{R}^{p-1}_s := \mathbb{R}^{p-1} \times \mathbb{R}^s_+ \to \mathbb{R}$ given by

$$(x_1, \ldots, x_{p-1}, y_1, \ldots y_s) \mapsto \underbrace{\frac{1}{2}\sum_{i=1}^{p-1} x_i^2}_{\text{hyperbolic part}} - \underbrace{\sum_{i=1}^{s} \lambda_{p+i}^{-1} \log(y_i)}_{\text{quasi-hyperbolic part}}$$

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- Let $\Omega_\lambda = \{(z, (x, y)) \in \mathbb{R} \times \mathbb{R}_s^{p-1} \mid z \geqslant f_\lambda(x, y)\}$ foliated by f_λ level sets

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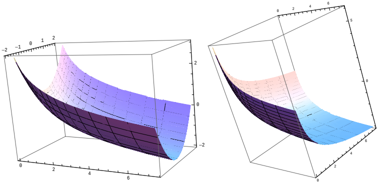


Figure: On the left $\Omega_{(0,0,1)}$ and on the right $\Omega_{(0,1,1)}$

Mixed cusps

Symmetry group

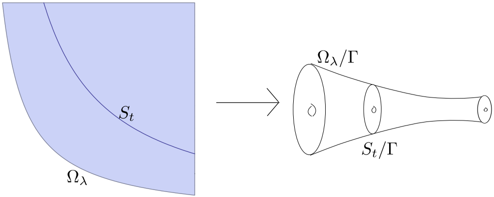
$$T_\lambda = \left\{ \begin{pmatrix} 1 & x & 0 & f(x,y) \\ 0 & I_{p-1} & 0 & x \\ 0 & 0 & D_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in PGL_{n+1}(\mathbb{R}) \mid (x,y) \in \mathbb{R}_s^{p-1} \right\}$$

$$O_\lambda = \underbrace{O_x}_{\text{Orthogonal}} \times \underbrace{P_{y,\lambda}}_{\text{Permutations}}$$

Diagonalizable cusps

Let $\lambda \in W_n$ with $\lambda_1 > 0$ and let
$O_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \lambda_i^{-1} \log(x_i) > 0\}$

O_λ is foliated by $S_t = \{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \lambda_i^{-1} \log(x_i) = t\}$



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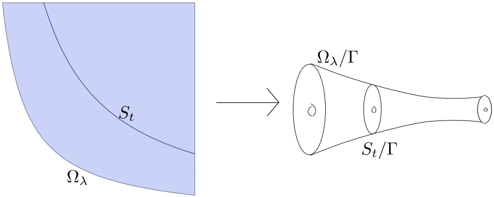
Let Γ be a lattice in the Lie group

$$T_\lambda = \left\{ \begin{pmatrix} u_1 & & & \\ & \ddots & & \\ & & u_n & \\ & & & 1 \end{pmatrix} \mid \sum_{i=1}^n \lambda_i^{-1} \log(u_i) = 0 \right\}$$

$O_\lambda =$ Coordinate permutation where $\lambda_i = \lambda_j$



Main Theorem

Theorem 1

(B–Cooper–Leitner) Let $C = \Omega/\Gamma$ *be an n-dimensional generalized cusp. Then there is a is a* $\lambda \in W_n$*, unique up to scaling, such that*

- Γ *is conjugate to a lattice* $\Gamma' \subset G_\lambda$
- C *deformation retracts onto a submanifold* $C' = \Omega'/\Gamma$ *that is projectively equivalent to* Ω_λ/Γ'*.*

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- Realization Problem: given a generalized cusp C, can you find an *interesting* properly convex manifold M with a cusp projectively equivalent to C?
- Can we use the geometry of generalized cusps to give coordinates on the space of convex projective structures on a fixed manifold? (Fenchel-Nielsen coordinates)

Thank you