Tutorial: Shape Analysis of Curves and Surfaces

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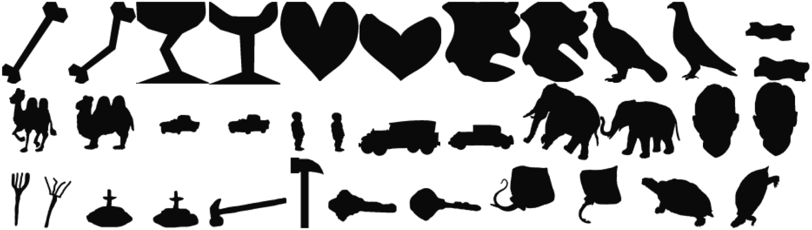
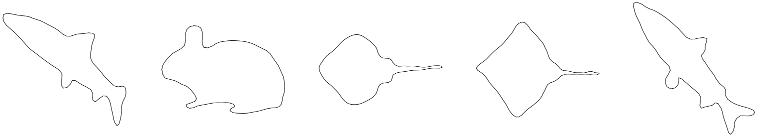
Functional Data and Shape Analysis

Approaches:

- Function space as a Riemannian manifold (Klassen, Charon, Le Brigant, Preston, Michor, Jermyn, Joshi, Moeller-Andersen, Needham, Harms,. . .)
- Functional data analysis via Riemannian metrics on the diffeomorphism group (Miller, Trouve, Zhang, Younes, Misiolek, Joshi,. . .)



Goals



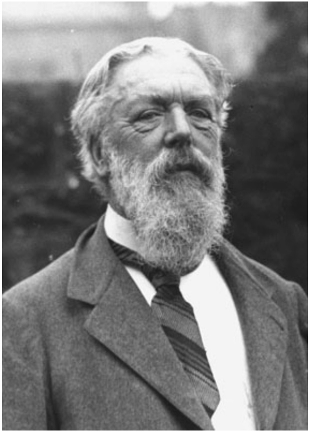
Goals

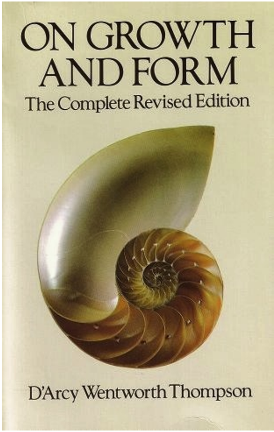
- Compare, describe and classify shapes
- Obtain a statistical framework for shape analysis

Difficulties:

- Spaces are inherently non-linear
- Spaces are high (infinite) dimensional

D'Arcy Thompson





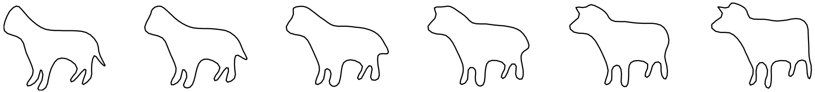
[www.darcythompson.org/catalogue_images/unframed/127.jpg]
[en.wikipedia.org/wiki/File:On_Growth_and_Form.JPG]

Shape analysis is analysis of deformations

In a very large part of morphology, our essential task lies in the comparison of related forms rather than in the precise definition of each; and the *deformation* of a complicated figure may be a phenomenon easy of comprehension, though the figure itself have to be left unanalysed and undefined. (. . .)

[D'Arcy Thompson. *On Growth and Form*. 1917]

A Riemannian setting for shape analysis



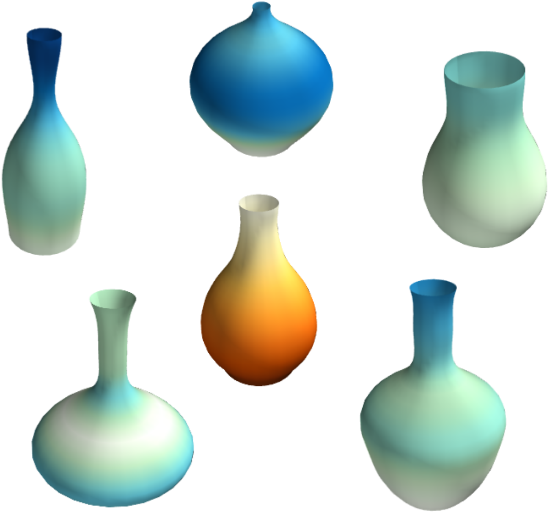
- Intuitive notion of similarity: Shapes that differ only by a small deformation are similar to each other.
- Gradients flows, geodesics, curvature.
- The exponential map may permit to linearize shape space.
- This eventually allows one to do statistics.

Infinte dimensional Riemannian geometry

Problems:

- Geodesic distance may vanish
- The metric might not be a bijection as a mapping from the tangent space to the cotangent space
- Existence of geodesic equation is not guaranteed
- Geodesic equations are PDEs: well-posedness, solutions, ...
- No theorem of Hopf Rinov
- The exponential map might not be a local diffeomorphism

Example: Shape Averaging



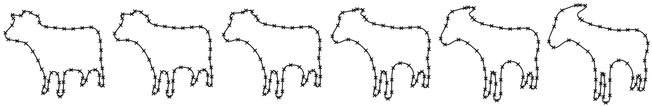
The manifolds of Curves and Surfaces

The space of curves is

$$\mathrm{Imm}(S^1, \mathbb{R}^2) = \{c \in C^\infty(S^1, \mathbb{R}^2) : c'(\theta) \neq 0\} \subset C^\infty(S^1, \mathbb{R}^2)\,.$$

More general: The space of surfaces of type M is

$$\mathrm{Imm}(M, \mathbb{R}^d) = \{f \in C^\infty(M, \mathbb{R}^d) : Tf \text{ is inj.}\} \subset C^\infty(M, \mathbb{R}^d)\,.$$



The Manifolds of Curves and Surfaces

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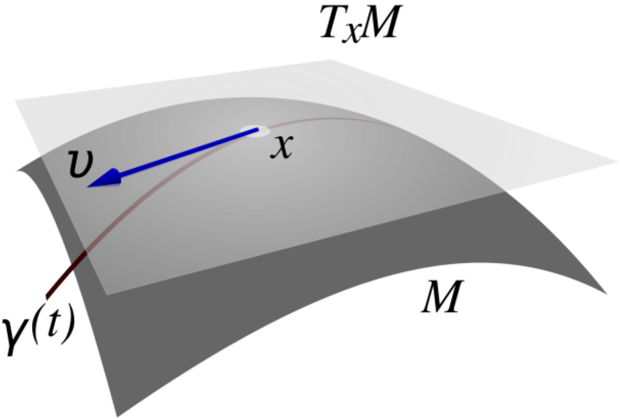

The Manifolds of Curves and Surfaces

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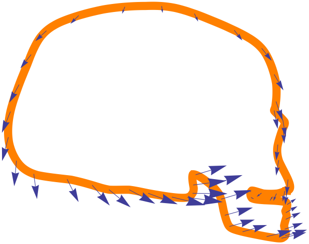
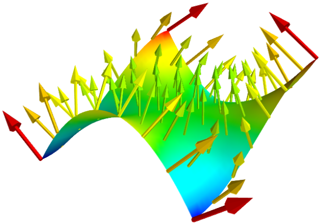
The tangent space of $\operatorname{Imm}(M, \mathbb{R}^d)$ at a curve/surface f is the set of all vector fields along f,

$$T_f \operatorname{Imm}(M, \mathbb{R}^d) = \left\{ h : \begin{array}{ccc} & & T\mathbb{R}^d \\ & \overset{h}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{f} & \mathbb{R}^d \end{array} \right\} \cong \left\{ h \in C^\infty(M, \mathbb{R}^d) \right\}\,.$$

Tangent vectors and paths on the manifold of curves





Diffeomorphism groups

The diffeomorphism group of the parameter space:

$$\operatorname{Diff}(M) = \{\varphi \in C^\infty(M, M) : \varphi \text{ bij.}\}$$

The diffeomorphism group of the ambient space (needs decay conditions):

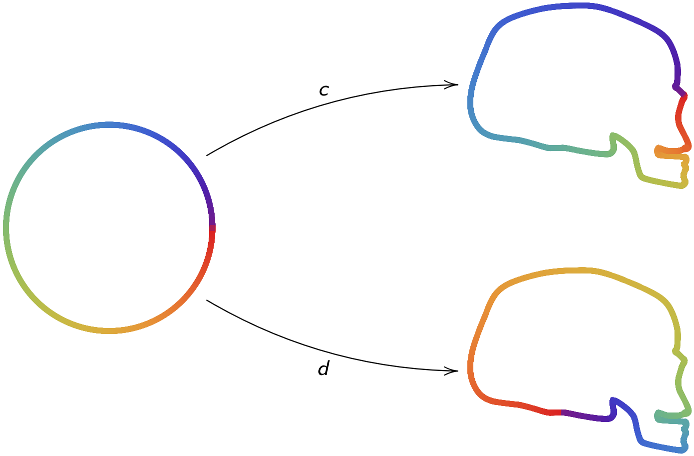
$$\operatorname{Diff}(\mathbb{R}^d) = \left\{\varphi \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) : \varphi \text{ gbij.}\right\}$$

Both of these groups are acting on Imm:

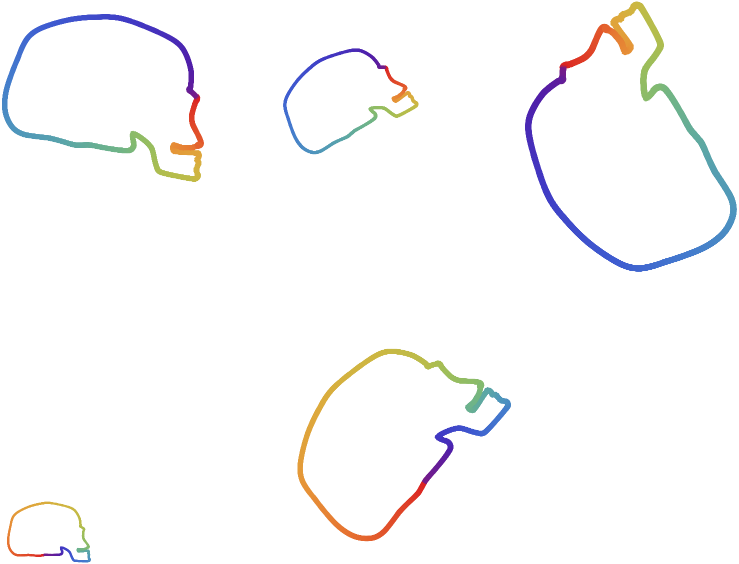
$$\operatorname{Diff}(\mathbb{R}^d) \to \operatorname{Imm}(M, \mathbb{R}^d) \leftarrow \operatorname{Diff}(M)\ .$$

Right action of Diff (Shape preserving)

$$(\mathrm{Imm}(M,\mathbb{R}^d),\mathrm{Diff}(M)) \to \mathrm{Imm}(M,\mathbb{R}^d)$$
$$(f,\varphi) \mapsto f\circ\varphi$$

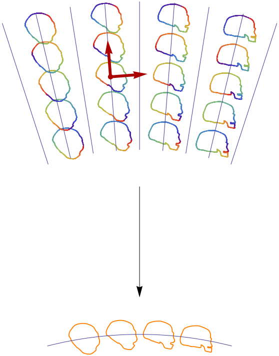


Other Shape preserving Transformations



Definition of shape space

Consider the group actions of reparametrizations Diff(M), Scalings (S), Translations (T), Rotations (R).



$$\mathrm{Imm}(M, \mathbb{R}^d)$$

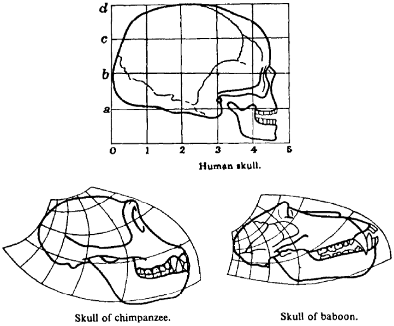
$$\downarrow$$

$$\mathrm{Imm}(M, \mathbb{R}^d)/\,\mathrm{Diff}(M)/R/S/T = \mathcal{S}(M, \mathbb{R}^d)$$

Extrinsic metrics

Left action of Diff (Shape changing)

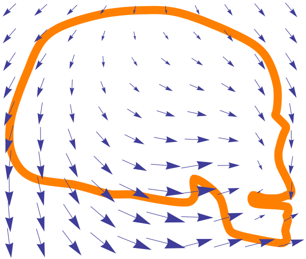
$$\begin{aligned}(\mathrm{Diff}(\mathbb{R}^d), \mathcal{S}(M, \mathbb{R}^d)) &\to \mathcal{S}(M, \mathbb{R}^d)\\ (\varphi, f) &\mapsto \varphi(f)\end{aligned}$$

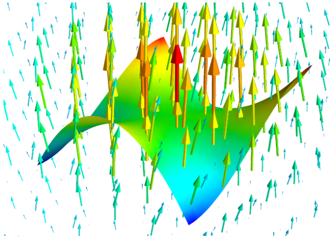


LDDMM: Measure cost of minimal transformation in $\mathrm{Diff}(\mathbb{R}^d)$ to define distance between curves/surfaces/images.

LDDMM (or extrinsic metrics)

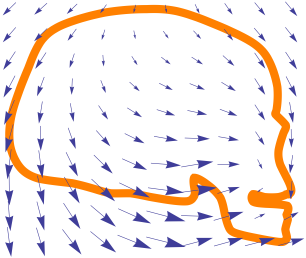
- Defines similarity between shapes via right invariant metric on Diffeomorphism group of ambient space.
- Works for all objects on which the Diffeomorphism group acts: curves, surfaces, images, densities, measures,...
- Minimization problem:

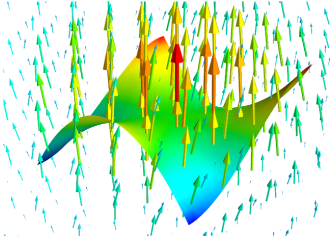




LDDMM (or extrinsic metrics)

$$\langle\langle X, Y \rangle\rangle = \int_{\mathbb{R}^d} \langle X, AY \rangle \, \mathrm{d}x$$





Matching problem:

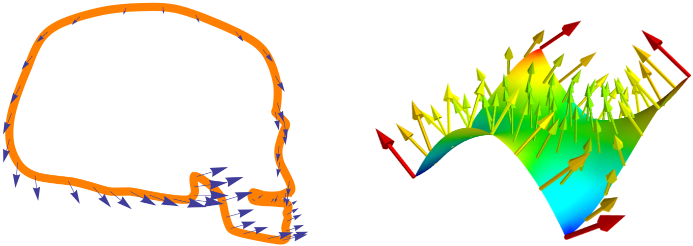
$$\operatorname{dist}([f_0], [f_1]) = \inf \int_0^1 \langle\langle X(t), X(t) \rangle\rangle dt$$

such that $\operatorname{Flow}(X)(f_0) = f_1 \circ \varphi$ for some $\varphi \in \operatorname{Diff}(M)$.

Intrinsic metrics

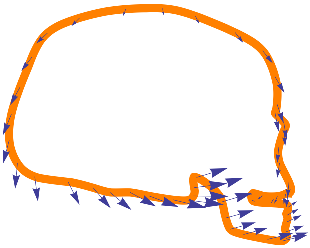
Intrinsic metrics

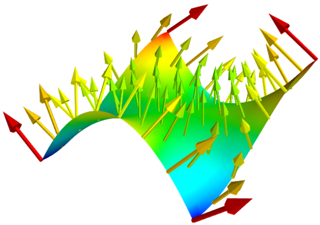
- Do not make use of this action but define a metric on the space Imm
- Minimization problem:



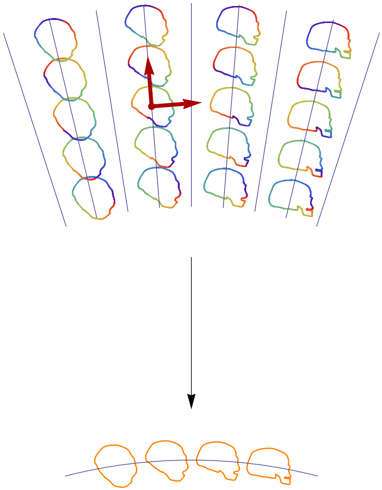
Defining an intrinsic metric

$$G_f(h, k) = \int_M \langle h, A_f k \rangle \,\mathrm{vol}$$





Invariance



$$\mathrm{Imm}(M, \mathbb{R}^d) \xrightarrow{\pi} \mathcal{S}(M, \mathbb{R}^d)$$

An invariant metric "above" induces a metric "below" such that π is a Riemannian submersion.

$$G_f(h, k) = G_{f\circ\varphi}(h \circ \varphi, k \circ \varphi)$$

Riemannian submersions

$$\begin{array}{c} \mathrm{Imm} \\ \Big\downarrow{\scriptstyle \pi} \\ \mathcal{S} \end{array}$$

- Horizontal geodesics on Imm project down to geodesics in shape space.
- Induced geodesic distance on quotient space:

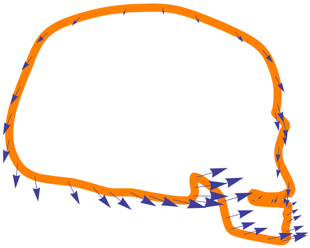
$$\mathrm{dist}^{\mathcal{S}}([f_0],[f_1]) = \inf_{\varphi} \mathrm{dist}(f_0, f_1 \circ \varphi)$$

- O'Neill's formula connects sectional curvature on Imm and on $\mathcal{S}$.

Defining an invariant metric (curves)

$$G_c(h,k) = \int_M \langle h, A_c k \rangle \,\mathrm{vol}$$

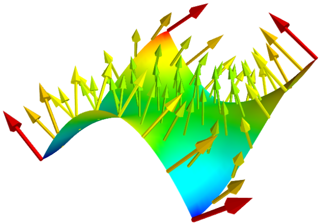
with $\mathrm{vol} = |c'|\,\mathrm{d}\theta$ and A_c defined in terms of $D_s = \frac{1}{|c'|}\partial_\theta$, e.g. $A_c = -D_s^2$.



Defining an invariant metric (surfaces)

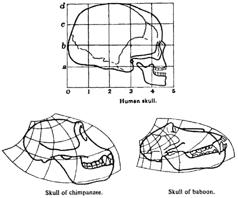
$$G_f(h,k) = \int_M \langle h, A_f k \rangle \,\mathrm{vol}$$

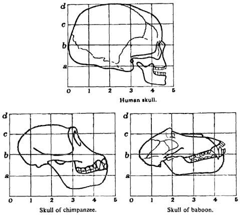
with vol being the surface volume form and A_f defined in terms of the surface Laplacian, e.g., $A_f = -\Delta_f$.



Differences between extrinsic metrics and intrinsic metrics

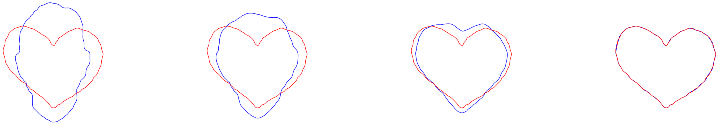
- Intrinsic metrics: Metric is defined on parameter space (lower dimensional), potentially computational faster
- Intrinsic metrics: Inertia operator depends highly on the foot point f, potentially more difficult (expensive) to implement
- Extrinsic metrics: yields deformtion of the ambient space in addition to registration.
- Intrinsic metrics: can create self intersections



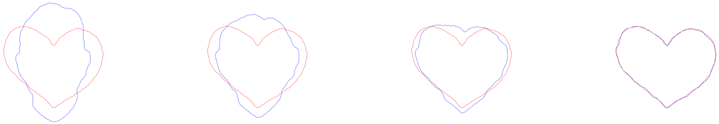


Example 1: Face to heart

Intrinsic:



Extrinsic:



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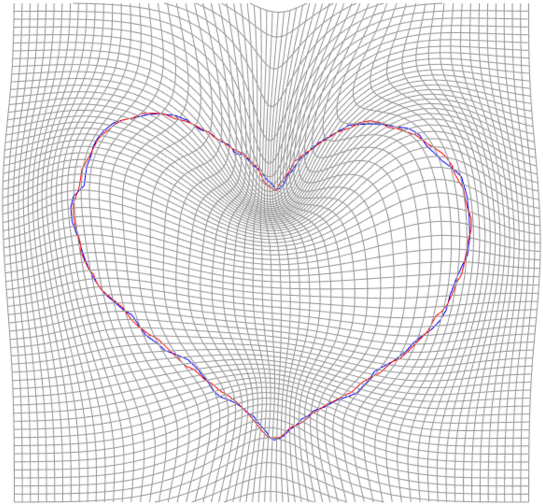


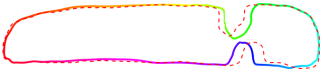
Example 2: Moving bumps

Intrinsic:

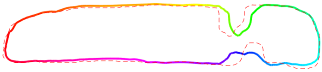
Extrinsic:

Example 2: Moving bumps

Intrinsic



Extrinsic



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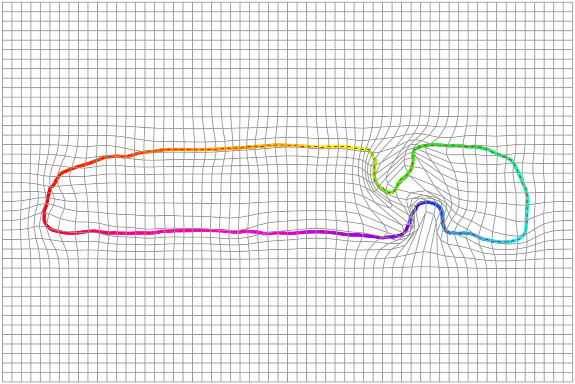


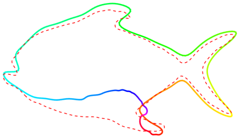
Example 3: Expanding a thin structure

Intrinsic:

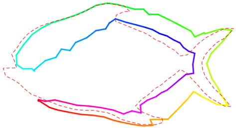
Extrinsic:

Example 3: Expanding a thin structure

Intrinsic



Extrinsic



Thank you