Lines in Space

This short section is about equations of lines in three dimensions. The first equation is analogous to the point-slope form in two dimensions. The second is analogous to the the two point form. The equation

\[ \mathbf{X} = \mathbf{X}_0 + \mathbf{D} t \]  

is the vector equation of the line going through the “point” or rather the vector \( \mathbf{X}_0 = (x_0, y_0, z_0) = x_0 \mathbf{i} + y_0 \mathbf{j} + z_0 \mathbf{k} \) and in the direction \( \mathbf{D} = (a, b, c) \). Equivalently, the vector equation above is equivalent to the three scalar equations

\[ x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \]  

These are sometimes called parametric equations, since the parameter \( t \) need not be one of the coordinate axes. The equation of the line through the vectors \( \mathbf{X}_0 \) and \( \mathbf{X}_1 \) can be obtained from equation 1 by letting \( \mathbf{D} = \mathbf{X}_1 - \mathbf{X}_0 \).

Note when \( t = 0 \) the curve is at the vector \( \mathbf{X}_0 \) or the point \((x_0, y_0, z_0)\) and in the second case the curve is at \( \mathbf{X}_1 \) when \( t = 1 \). Some example problems. Maple will plot lines, and more general parametric equations with the spacecurve command (needs with(plots);), for example the command below will plot the parametric equations \( x = \sin t, \quad y = \cos t, \quad z = 2 \pi t \) which is the graph of a helix.

\[ \text{spacecurve}([\sin(t), \cos(t), 2 \pi t], t = 0..3, \text{title}= '3 turns of a Helix'); \]

\#1 Find the vector equation of the line through the points \((1, 0, 1)\) and \((3, 2, 0)\).

Ans. \( \mathbf{D} = (3 - 1, 2 - 0, 0 - 1) = (2, 2, -1) \). So \( \mathbf{X} = (1, 0, 1) + \mathbf{D} t = (1 + 2t, 2t, 1 - t) \).

\#2 Find where the line \( x = 1 + t, \quad y = t, \quad z = 1 - t \) intersects the sphere \( x^2 + y^2 + z^2 = 14 \).

Ans. Substituting in we have

\[
\begin{align*}
(1 + t)^2 + t^2 + (1 - t)^2 &= 14 \\
1 + 2t + t^2 + t^2 + 1 - 2t + t^2 &= 14 \\
2 + 3t^2 &= 14 \\
t^2 &= 4 \\
t &= -2 \quad \text{or} \quad 2
\end{align*}
\]

So when \( t = 2 \) we have the point of intersection \((3, 2, -1)\) and when \( t = -2 \) we have the point of intersection \((-1, -2, 3)\).

\#3 Find where the two lines \( x = 1 + t, \quad y = 2 + t, \quad z = 3 + t \) and \( x = 1 + 3s, \quad y = 1 + 4s, \quad z = 1 + 5s \) intersect.

Ans. Note that not every pair of lines intersect. Parallel lines in the plane to do not intersect, in space there are non-parallel lines (skew lines) which do not intersect. The solution is to solve the equations the three equations \( 1 + t = 1 + 3s, 2 + t = 1 + 4s, 3 + t = 1 + 5s \) in two unknowns. If it has no solution the lines do not intersect, if it has infinitely many solutions, the lines are the same, and if it has a unique solution it will give the point of intersection. In practice, if is often fastest to solve two of equations say \( 1 + t = 1 + 3s, 2 + t = 1 + 4s \) and substitute in the third to see if it is a solution to the third as well. Here if we subtract equation 1 from 2 we get \( 1 = s \) and so \( 1 + t = 1 + 3(1) \) or \( t = 3 \). In this case \( t = 3, s = 1 \) is a solution to the third equation as well so \((1 + 3, 2 + 3, 3 + 3) = (4, 5, 6) = (1 + 3(1), 1 + 4(1), 1 + 5(1))\) is the point of intersection. Note if \( z = 1 + 5s \) is replaced by \( z = 0 + 5s \), then the lines do not intersect.

\#4 Find the equation of a line through \((1, 1, 1)\) parallel to \( \mathbf{X} = (3 - t, 2 - 2t, 5t) \).

Ans. The lines having the same direction are parallel. So we can use the direction \( \mathbf{D} = (-1, -2, 5) \), so \( \mathbf{X} = (1, 1, 1) + (-1, -2, 5) t \) is a solution.

\#5 Find the equation of line which is the intersection of the two planes \( x + y + z = 3 \) and \( x - y + z = 5 \)

Ans. Solve for two variables in terms of the third. Solving \( x + y = 3 - z, x - y = 5 - z \) yields \( x = 4 - z, y = -1 \) which yield the parametric equations \( x = 4 - z, y = -1, z = z \) or perhaps the equivalent form \( x = 4 - t, y = -1 + 0t, z = 0 + t \) looks more satisfying.

\#6 Find the equation of the y-axis.

Ans. A point on the line is \((0, 0, 0)\) and the direction is \( \mathbf{j} = (0, 1, 0) \) which yields the equations \( x = 0, y = t, z = 0 \)

Exercise: Find the coordinates of the point on the plane \( x + 2y + 4z = 4 \) nearest the origin. Repeat for the plane \( Ax + By + Cz = D \).