The Wrath of Khan

Beyond the darkness, beyond the human evolution is Khan, a genetically superior tryant, exiled to a barren planet by a starship commander he is destined to destroy, left for dead, he has survived and now escaped. Quoting the Klingons, "Revenge is a dish best served cold" he waits in ambush.

Meanwhile, the starship commander has been promoted to a desk job, but is out in the field again as an observer on a cadet training mission in his old ship. Simulations of classic problems follow one after another. Many of the old crew are still at hand and a familar voice comes from engineering.

"I've given her all she's got captain, and I can't give her no more. ... She can't take much more, Captain, she's gonna blow! ... Aye cannot change the laws the laws of physics." For once there is no rush so the question is asked, "What laws of physics Scotty?"

"It's the heat equation sir. We can model the dilithum crystals as a rod one unit long with both ends kept at fixed temperatures. The rod heats up and the problem is that it has to cool according to the PDE and boundry conditions given below. $u(x, t)$ is the temperature x units from the left end point of the rod at time t.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{PDE}$$

$$u(0, t) = 0, \text{ and } u(1, t) = 0 \text{ for all } t \geq 0 \tag{BC}$$

$$u(x, 0) = f(x) \text{ for all } 0 \leq x \leq 1 \tag{IC}$$

The initial temperature $u(x, 0)$ for $0 < x < 1$ is given (IC). The ends temperatures $u(0, t)$ and $u(1, t)$ are fixed for all time t (BC). At the other points (x, t) with $0 < x < 1$ and $0 < t$ the temperature is determined by the partial differential equation (PDE)".

"But Scotty, I don't know how to solve these equations. In fact the only differential equations I know I got in Calculus 2."

"Don't worry sir. You don't need to know how to solve them in order to understand. I can give you the solutions and you can check that they are solutions. In fact, we solve them by a simple numerical simulation in engineering sir."

a. Let $u(x, t)$ be a solution to the PDE. At each point (x_0, t_0) with $0 < x_0 < 1$ and $0 < t_0$ the PDE implies that the **concavity** of the function $f(x) = u(x, t_0)$ at $x = x_0$ is related to the **monotonicity** of the function $g(t) = u(x_0, t)$ at $t = t_0$. (Monotonicity means increasing or decreasing.) Find and state this relationship. Further, suppose for our solution u, $f(x) = x(x - \frac{1}{3})(x - \frac{2}{3})(x - 1)$. Find the regions on $[0, 1]$ where f is positive and negative and the regions where the solutions u are increasing and decreasing functions of t. (Note the regions are different. In particular, there are points (x_0, t_0) where $u(x_0, t_0)$ is negative, but where the temperature is decreasing [as t increases].)

b. Show the functions below for integers $n > 0$ solve the PDE and the boundry condition BC.

$$u(x, t) = \sin(nx\pi)e^{-n^2\pi^2 t} = \mathbf{U_n}$$

c. Show that if $u(x, t)$ and $v(x, t)$ are both solutions to the PDE and the BC, then so are the functions $u(x, t) + v(x, t)$ and $cu(x, t)$ for any constant c. Problems with this property are called linear. Show that $U_1 - U_3/3^3 + U_5/5^2 - U_7/7^2$ is a solution where U_n is defined in part b. [This "series" for $t = 0$ converges to the "triangle function", $f(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $f(x) = 2 - 2x$ for $\frac{1}{2} \leq x \leq 1$, but you don't have to show it.]

d. Show if n is an odd integer, then average value of $n * \sin(n * \pi x)$ over $0 \leq x \leq 1$ is $2/\pi$. Hence for each odd n, the average temperature of nU_n is the same at $t = 0$. (U_n as in part b.). Furthermore, show nU_n has a maximum value of n. It turns out that as n increases, U_n decays much faster. Illustrate this with a single plot of the three functions U_1, $3U_3$ and $5U_5$ over the ranges $0 \leq x \leq 1$, $0 \leq t \leq 0.03$.

Now we turn to the numerical approximation. This requires using a spreadsheet or Maple matrices to hold the data. An example for the wave equation was done in class on Halloween.

e. Derive the difference equation by using the one sided estimate for u_t and the two-sided estimate for u_x and u_{xx}

$$u(x, t + \Delta t) = u(x, t) + (\Delta t/\Delta x^2)(u(x - \Delta x, t) - 2u(x, t) + u(x + \Delta x, t))$$

f. Using $\Delta x = 0.05$ (21 data points from 0 to 1) and $\Delta t = 0.001$ (101 data points from 0 to 0.1). Construct a numerical solution to the heat equation using the initial values of U_1, namely $\sin(x * \pi)$. And plot your numerical solution. [It is important that Δt is small compared to Δx.]

g. Compare your numerical answer with the exact answer U_1 on your matrix of data points. [You need to construct a matrix for the exact solution too.] Plot the matrix difference, approximate numerical solution − exact solution. Find the largest error and the largest percentage error. Does the plot make sense?

h. Use your numerical solver to compare two solutions. The first has initial temperature 1 for $0 < x \leq 0.5$ and 0 otherwise. The second has initial temperature 1 for $0.5 \leq x < 1$ and 0 otherwise. Plot the two numerical solutions together. Note how similar they are at time $t = 0.1$.

We are done with the project statement. But what about Khan you ask? Well his ambush was sucessful, but not conclusive. A rematch found the starships fighting in a nebula with limited visibility. Khan is intellegent, but not experienced, and his pattern of fighting indicated only two dimensional thinking. [Try changing $\Delta t = 0.01$ in part f to see what happened. (This is a numerical instability, it is not "real".)] Of course, the words "Scotty, I need warp speed in 3 minutes or we're all dead," are part of the climax.

From the Course Syllabus

PROJECT. You will work on the project in groups of 1–4 students. This project will be a substantial assignment, giving you a chance to earn part of your grade in an environment which simulates the so-called "real world" better than does an in-class exam. It will also give your instructor a chance to base part of your grade on your best work, produced in a setting where time should not be a factor (assuming you start on your project as soon as it is assigned). The results of your work on your project will be presented in a report (one report per group). Each member will also submit a "group evaluation" giving their impression of the relative contribution of each member to the group's effort. These evaluations are due with the project. It is not guaranteed that each member of the group will receive the same grade. The reports will be graded not only on their mathematical content but also on the quality of the presentation: clarity, neatness, and proper grammar are also important. Both reports and group evaluations must be typed. The project will be assigned on (Tentatively) Thursday, October 31 and due on Thursday, November 14.

Grading

Each part is worth 10 points which leaves 20 points for clarity, neatness, grammar and general wow value, for a total of 100 points. You are free to use Maple on all the calculations, but for clarity you should do the easy ones by hand. There are a very small number of bonus points.

If you like the statement of this project, you might also like:
http://us.imdb.com/Title?0084726
Also some maple drawn graphics illustrating some of objects in the project is available at:
http://www.math.fsu.edu/~bellenot/class/f02/cal3/project.html