Problems on Finite Dimensional Norms (part of HW#5)

1. Suppose \( p(x) \) and \( q(x) \) are two norms on \( \mathbb{R}^1 \).

   (a) Show there are positive finite constants \( A \) and \( B \) so that

   \[
   Aq(x) \leq p(x) \leq Bq(x)
   \]

   is true for all \( x \in \mathbb{R}^1 \)

   (b) Using (\( \ast \)), show \( p(\cdot) \) and \( q(\cdot) \) have the same Cauchy sequences and the same convergent sequences

   (and hence have the same topology).

   (c) Using (b) prove each one dimensional subspace \( G \), of any normed space \( E \) is closed.

2. Suppose \( \| \cdot \| \) is some norm on \( \mathbb{R}^2 \) and \( f : \mathbb{R}^2 \to \mathbb{R} \) is some linear functional. Let \( e_1 = (1,0) \), \( e_2 = (0,1) \), \( f_1(x,y) = x \) and \( f_2(x,y) = y \) be the usual vectors and functionals. Define candidates for norms

   \[
   \begin{align*}
   p((x_1,x_2)) &= \sum_i |x_i| \|e_i\| \\
   q((x_1,x_2)) &= \max_i \{|x_i|/\|f_i\|\}
   \end{align*}
   \]

   (a) Use (1c) to show \( f \) is continuous. (And hence \( q(\cdot) \) is well-defined.)

   (b) Show \( p(\cdot) \) and \( q(\cdot) \) are norms and for all \( (x_1,x_2) \),

   \[
   q((x_1,x_2)) \leq \| (x_1,x_2) \| \leq p((x_1,x_2))
   \]

   (c) Find \( 0 < M < \infty \) so that for all \( (x_1,x_2) \)

   \[
   p((x_1,x_2)) \leq Mq((x_1,x_2))
   \]

   (d) Show there are positive finite constants \( A \) and \( B \) so that \( A\|x\| \leq \sum |x_i| \leq B\|x\| \) holds for all \( x \in \mathbb{R}^2 \)

   (e) Repeat (1b) and (1c) to prove each two dimensional subspace \( G \), of any normed space \( E \) is closed.

3. Suppose \( G \) is a codimension one closed subspace of the normed space \( E \) and \( x_0 \not\in G \). Then \( E = \{tx_0 + y : t \in \mathbb{R}, y \in G\} \). Let \( \phi : E \to \mathbb{R} \) be given by \( \phi(tx_0 + y) = t \). Compute the norm of \( \phi \) in terms of the the distance from \( x_0 \) to \( G \) which is \( \inf\{\|x_0 - g\| : g \in G\} \).