Problems on continuous linear functionals (part of HW#4)

1. Let $E = \mathbb{R}^2$ with the norm $\| (x, y) \| = \max \{|x|, |y|, |y - x|\}$
   
   (a) Draw the unit sphere $S = \{(x, y) : \| (x, y) \| = 1 \}$
   
   (b) Compute the norms of the functionals $f(x, y) = x$, $g(x, y) = y$, $h(x, y) = x + y$ and $k(x, y) = x - y$.
   
   (c) Derive a formula for the norm of the function $f(x, y) = ax + by$ for any reals $a$ and $b$.

2. Let $c = \{(x_n) : \lim x_n \text{ exists}\}$ with the sup norm so that $c_0 \subset c \subset m = \ell_\infty$.
   
   (a) Show $\phi : c \to \mathbb{R}$ given by $\phi((x_n)) = \lim x_n$ is a continuous linear function on $c$ and compute its norm.
   
   (b) show $\phi$ is not defined on $m$ and $\phi$ is the 0 functional on $c_0$.
   
   (c) Let $\eta$ be the sequence (1) which is constantly one. Show for $x = (x_n) \in c$, $x - \phi(x)\eta \in c_0$.

3. Let $c$ and $\phi$ be as in 2. Show the dual of $c$ is given by
   
   $c^* = \{\alpha \phi + x : \alpha \in \mathbb{R}, x \in \ell_1\}$
   
   Part of this problem is coming up with a formula for $\| \alpha \phi + x \|$. Hint: If $y = (y_n) \in \ell_1$ and $x = (x_n) \in c$ then $\langle y, x \rangle = \sum_n y_n x_n$ is the linear functional on $c$ associated with this $y$ in $\ell_1$. 