Problems on Generalized Functions (part of HW#7)

1. Let \( f_n(x) = n \chi_{[1/n, 2/n]} \) and \( T_n \) the distribution associated with \( f_n \).
   
   (a) Show \( T_n \to \delta \).
   
   (b) Find the derivatives of \( T_n \).
   
   (c) Show \( T'_n \to \delta' \).

2. Show
   
   (a) \( \delta(ax) = \delta(x)/|a| \)
   
   (b) \( f(x)\delta(x) = f(0)\delta(x) \)
   
   (c) \( f(x)\delta'(x) = -f'(0)\delta(x) \)

3. Give an example of two sequences of functions \((a_n)\) and \((b_n)\) with associated distributions \((\alpha_n)\) and \((\beta_n)\) so that both sequences converge to the distribution \( \delta \), but so that \( \langle T, \phi \rangle = \lim_n \int_{-\infty}^{\infty} a_n(t)b_n(t)\phi(t)\,dt \) converges but to something different than \( \phi(0)^2 \).