The last problem set

1. Let $E$ be a norm space with dual $E^*$, if $A \subset E$ (resp $B \subset E^*$) define $A^\perp = \{ f \in E^* : f(a) = 0 \ \forall a \in A \}$ (resp $B^\top = \{ x \in E : f(x) = 0 \ \forall f \in B \}$)

   (a) Show both $A^\perp$ and $B^\top$ are closed subspaces.
   (b) Show $A \subset A^{\perp \top}$ and $B \subset B^{\top \perp}$
   (c) Show $A^{\perp \top}$ is the closure of the linear span of $A$.
   (d) If $E = \ell_1$ and $B = c_0 \subset m = \ell_\infty = E^*$ is a closed subspace of $E^*$, but $B \neq B^{\top \perp}$

2. Show if $T : E \to E$ is bounded linear operator, then let $N_n = \{ x \in E : T^n x = 0 \}$ be the kernel of $T^n$. Show $N_0 \subset N_1 \subset N_2 \cdots \subset N_n \subset N_{n+1} \cdots$

   (a) Show if $A$ is compact then for $T = I - A$ there is an $m$ so that for all $k$, $N_m = N_{m+k}$.
   (b) Show that for the shift to the left $T : \ell_2 \to \ell_2$ given by $T((a_1, a_2, a_3, \ldots)) = (a_2, a_3, a_4, \ldots)$ the subspaces satisfy $N_n \neq N_{n+1}$

3. Show if $(x_n)$ are norm one elements of the norm space $E$, $(f_n)$ are norm one functionals in $E^*$, and $(\lambda_n)$ are scalars with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ then the operator $A(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) x_n$ is a well-defined compact operator on $E$.

   Suppose further that $f_i(x_j) = \delta_{i,j}$, show that $x_n$ is an eigenvector of $T$ with eigenvalue $\lambda_n$.

4. If $A : E \to E$ compact then for every $x_n$ with $x_n \to x$ weakly, then $Ax_n \to Ax$ in norm.

   If $E$ is reflexive, then the converse is also true.