1 Separation of Variables

Find the solution $u(x, y)$ to the following equations by separating variables.

1. $u_x + u_y = 0$  
   answer: $u = ce^{k(x+y)}$

2. $u_x - u_y = 0$  
   answer: $u = c \exp \left[ \frac{1}{2}(x^2 + y^2) + k(x - y) \right]$

3. $y^2 u_x - x^2 u_y = 0$

4. $u_x + u_y = (x + y)u$  
   answer: $u = c \exp(kx + y/k)$

5. $u_{xx} + u_{yy} = 0$

6. $u_{xy} - u = 0$  
   answer: $u = x^k e^{-y^2/k}$

7. $x u_{xy} - 2yu = 0$

The next group of problems are boundary value problems

9. $X'' = \lambda X; X(0) = X(L) = 0$  
   answer $\lambda = -w^2 = -(n\pi/L)$ and $X(x) = C \sin(n\pi x/L)$

10. $X'' = \lambda X; X(0) = X'(L) = 0$

11. $X'' = \lambda X; X'(0) = X'(L) = 0$  
    answer $\lambda = -w^2 = -(n\pi/L)^2$ and $X(x) = C \cos(n\pi x/L)$ or $\lambda = 0$ and $X(x) = C$

12. $X'' = \lambda X; X(0) = X(L); X'(0) = X'(L)$

    Solution to #4 above. Let $u = X(x)Y(y)$, plugging to the equation gives

    $$X'(x)Y(y) + X(x)Y'(y) = (x + y)X(x)Y(y)$$

    $$\frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = (x + y)$$

    $$\frac{X'(x)}{X(x)} - x = k = y - \frac{Y'(y)}{Y(y)}$$

    for some constant $k$. We have two ODE to solve

    $$X'(x) - (x + k)X(x) = 0$$  and  $$Y'(y) - (y - k)Y(y) = 0$$

    The first has an integrating factor of $\exp(-x^2/2 - kx)$ and solution $X(x) = C \exp(x^2/2 + kx)$. The second has an integrating factor of $\exp(-y^2/2 + ky)$ and solution $Y(y) = C \exp(y^2/2 - ky)$. Multiplying the ODE solutions gives the answer above.

    Solution to #7. $u = X(x)Y(y)$

    $$X''(x)Y(y) - X(x)Y''(y) = 0$$
\[
\frac{X''(x)}{X(x)} = k = \frac{Y''(y)}{Y(y)}
\]

\[
X''(x) - kX(x) = 0 \quad Y''(y) - kY(y) = 0
\]

Supposing \(k \neq 0\), we get \(X(x) = C_1 e^{wx} + C_2 e^{-wx}\) and \(Y(y) = C_1 e^{wy} + C_2 e^{-wy}\), where \(\omega\) is the (possibly complex) number so that \(\omega^2 = k\). Our answer has 4 terms

\[
u = A \exp(\omega(x + y)) + B \exp(\omega(x - y)) + C \exp(\omega(y - x)) + D \exp(-\omega(x + y))
\]

If \(k < 0\) and changing \(\omega\) so that \(-\omega^2\) we have the alternate solution \(X(x) = C_1 \cos \omega x + C_2 \sin \omega y\) and \(Y(y) = C_1 \cos \omega y + C_2 \sin \omega y\) Our answer has four different terms

\[
u = A \cos \omega x \cos \omega y + B \cos \omega x \sin \omega y + C \sin \omega x \cos \omega y + D \sin \omega x \sin \omega y
\]

Finally if \(k = 0\), \(X(x) = C_1 x + C_2\) and \(Y(y) = C_1 y + C_2\) giving the solution

\[
u = Axy + Bx + Cy + D
\]

## 2 Characteristic examples, Normal form table

If the PDE is \(au_{xx} + bu_{xy} + cu_{yy} = 0\) and the roots of \(ax^2 - bx + c\) are \(r\) and \(s\). (Note the change of sign if \(b\) in the PDE to \(-b\) in the polynomial.) The constant coefficient case looks like:

<table>
<thead>
<tr>
<th>Type</th>
<th>Hyperbolic</th>
<th>Parabolic</th>
<th>Elliptic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roots (r) and (s)</td>
<td>real and (r \neq s)</td>
<td>real and (r = s)</td>
<td>complex (r = a + bi), (s = a - bi)</td>
</tr>
<tr>
<td>Characteristics</td>
<td>(\Phi = y - rx, \Psi = y - sx)</td>
<td>(\Phi = \Psi = y - rx)</td>
<td>(\Phi = y - rx, \Psi = y - sx)</td>
</tr>
<tr>
<td>New variables</td>
<td>(\xi = y - rx, \eta = y - sx)</td>
<td>(\xi = x, \eta = y - rx)</td>
<td>(\xi = y - ax, \eta = bx)</td>
</tr>
<tr>
<td>Solution</td>
<td>(u = f(y - rx) + g(y - sx))</td>
<td>(u = f(y - rx) + xg(y - rx))</td>
<td>(u = f(y - rx) + g(y - sx))</td>
</tr>
<tr>
<td>Normal form</td>
<td>(u_{\xi \eta} = 0) or (u_{\xi \xi} - u_{\eta \eta} = 0)</td>
<td>(u_{\eta \eta} = 0)</td>
<td>(u_{\xi \xi} + u_{\eta \eta} = 0)</td>
</tr>
</tbody>
</table>

Some motivation for why this works.

Of course the most interesting question is why the sign change? It is not hard to check that \(ax^2 + bx + c\) and \(ax^2 - bx + c\) have the roots that are negative of each other. So if \(r\) and \(s\) are roots of \(ax^2 - bx + c\) then \(-r\) and \(-s\) are roots of \(ax^2 + bx + c\). Eventually this means \(ax^2 + bx + c = a(x + r)(x + s)\). Symbolically we can write

\[
a \left( \frac{\partial}{\partial x} + r \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + s \frac{\partial}{\partial y} \right) u = au_{xx} + bu_{xy} + cu_{yy} = 0
\]

If you look at \(u_x + ru_y = 0\), this says that the directional derivation of \(u\) in the \((1, r)\) direction is always zero. So \(u\) is constant along lines perpendicular to \((-r, 1)\), that is \(u\) is constant on lines of the form \(y - rx = C\) for some constant \(C\). This change of sign reflects the change from the direction to the normal direction.

## 3 Characteristic examples, Normal form problems

- We do the wave equation first \(c^2 u_{xx} - u_{yy} = 0\). Step 1: \(A = c^2\), \(B = 0\), \(C = -1\) and thus \(AC - B^2 = -c^2 < 0\) so the equation is hyperbolic.

  Step 2: is the find the characteristics, we need to solve

\[
A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0
\]

\[
c^2 \left( \frac{dy}{dx} \right)^2 - 1 = 0
\]

\[
\frac{dy}{dx} = \pm 1/c
\]
Which gives \( y = x/c + C \) and \( y = -x/c + C \) so \( \Phi = x - cy \) and \( \Psi = x + cy \) are the characteristics.

Step 3: We solve the equation as \( u = f(x - cy) + g(x + cy) \) Check that it solves the equation.

Step 4: Transforms \( \xi = x - cy \) and \( \eta = x + cy \) gives \( u_x = u_\xi + u_\eta, \ u_y = -cu_\xi + cu_\eta, \ u_{xx} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta}, \ u_{yy} = c^2u_{\xi\xi} - c^2u_{\xi\eta} - c^2u_{\eta\xi} + c^2u_{\eta\eta}, \) So
\[
e^2u_{xx} - u_{yy} = 4c^2u_{\xi\eta}
\]

and the equation has the canonical form \( u_{\eta\eta} = 0 \)

- Problem #13 in §12.4 gives the PDE \( u_{xx} + 9u_{yy} \) and asks us to find the type, transform to normal form and solve. Step 1 is to classify the equation, clearly \( A = 1, B = 0 \) and \( C = 9 \) so that \( AC - B^2 = 9 > 0 \) and the equation is elliptic.

Step 2 is to find the characteristics, we need to solve
\[
A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0
\]
\[
\left( \frac{dy}{dx} \right)^2 + 9 = 0
\]
\[
\frac{dy}{dx} = \pm 3i
\]
Which gives \( y = 3ix \) and \( y = -3ix \), we write these as \( \Phi = y - 3ix \) and \( \Psi = y + 3ix \) as characteristics.

Step 3 from the characteristics, we can solve the equation as
\[
u(x, y) = f(y - 3ix) + g(y + 3ix)
\]
Note assuming complex variables behave
\[
u_{xx} = (-3i)^2f''(y - 3ix) + (3i)^2g''(y + 3ix) = -9f'' - 9g''
\]
\[
u_{yy} = f''(y - 3ix) + g''(y + 3ix) = f'' + g''
\]
and clearly \( \nu_{xx} + 9\nu_{yy} = 0. \)

Step 4, we use the transformations \( \xi = (\Phi + \Psi)/2 = y \) and \( \eta = (\Phi - \Psi)/2i = 3x \) to change the PDE to the canonical form \( u_{\xi\xi} + u_{\eta\eta} = 0. \) Eventually \( u_{\xi\xi} = u_{\eta\eta} \) and \( 9u_{\eta\eta} = u_{xx}. \) The change rule was use in step 4.
\[
u_x = u_\xi \xi_x + u_\eta \eta_x = 0u_\xi + 3u_\eta = 3u_\eta
\]
\[
u_{xx} = 3(u_\eta \xi_x + u_\eta \eta_x) = 9u_{\eta\eta}
\]
- Problem #15 \( u_{xx} + 2u_{xy} + u_{yy} = 0 \) Step 1 \( A = B = C = 1, \) so that \( AC - B^2 = 0 \) and the equation is parabolic.

Step 2:
\[
A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0
\]
\[
\left( \frac{dy}{dx} \right)^2 - 2 \frac{dy}{dx} + 1 = 0
\]
factors to \((\frac{dy}{dx}) - 1)^2 = 0\) and there is the one solution \( y = x + C \) so \( \Phi = (y - x) \) is a characteristic

Step 3: We need two equations, the second is \( x \) times something similar to the first so \( u = f(y - x) + xg(y - x) \) Lets check it \( u_x = -f'(y - x) + g(y - x) - xg'(y - x), \ u_y = f'(y - x) + xg'(y - x), \)
Step 2: gives a general solution of interchanged problem as
\[ u = \frac{1}{1} x \]

Step 4: Let \( \xi = y - x \) and \( \eta = x \) then \( u_x = -u_\xi + u_\eta, u_y = u_\xi + 0u_\eta, \)
\[ u_{xx} = -(-u_\xi + u_\eta) + (u_\eta + u_\xi) = u_\xi - 2u_\xi + u_\eta \]
\[ u_{xy} = -(u_\xi + 0u_\eta) + (u_\eta + 0u_\xi) = -2u_\xi + u_\eta \]
\[ u_{yy} = u_\xi + 0u_\xi = u_\xi \]
\[ u_{xx} + 2u_{xy} + u_{yy} = (1 - 2 + 1)u_\xi + 2(-1 + 1 + 0)u_\eta + (1 + 0 + 0)u_\eta = u_\eta \]

And so the canonical form is \( u_\eta = 0. \)

- Problem \#11 Requires a trick not discussed the text. Our PDE is \( u_{xy} - u_{yy} = 0. \) Step 1 \( A = 0, B = 1/2 \) and \( C = -1 \) so \( AC - B^2 = -1/4 < 0 \) and the equation is hyperbolic.

Step 2:
\[ A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0 \]
\[ 0 \left( \frac{dy}{dx} \right)^2 - \frac{dy}{dx} = 1 = 0 \]

Doesn’t have two solutions; What to do?

The trick is to interchange the variables. Solve the problem and interchange back. So solving \( uyx - u_{xx} = 0. \) Step 1: \( A = -1, B = 1/2 \) and \( C = 0, \) so \( AC - B^2 = -1/4 < 0 \) and the equation is hyperbolic.

Step 2:
\[ A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0 \]
\[ -1 \left( \frac{dy}{dx} \right)^2 - \frac{dy}{dx} = 0 \]

This factors into
\[ -\frac{dy}{dx} \left( \frac{dy}{dx} + 1 \right) = 0 \]

The first has solution \( y = C, \) so \( \Phi = y \) and the second has solution \( y = -x + C \) so \( \Psi = y + x. \) This gives a general solution of interchanged problem as
\[ u(x, y) = f(y) + g(y + x) \]

and so the non-interchanged problem should have \( \Phi = x, \Psi = x + y \) and general solution
\[ u(x, y) = f(x) + g(x + y) \]

Checking \( u_x = f'(x) + g'(x + y), u_{xy} = g''(x + y), u_y = g'(x + y) \) and \( u_{yy} = g''(x + y) \) so that
\[ u_{xx} - u_{yy} = 0. \]

- Problem \#19 Requires more steps than are in the text. It gives the PDE \( xu_{xx} - yu_{xy} = 0. \) Step 1 has \( A = x, B = -y/2 \) and \( C = 0, \) so that \( AC - B^2 = -y^2/4 < 0 \) (if \( y \neq 0 \)) and the equation is hyperbolic.

Step 2:
\[ A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0 \]
\[ x \left( \frac{dy}{dx} \right)^2 + y \frac{dy}{dx} = 0 \]

This factors into
\[ \frac{dy}{dx} \left( x \frac{dy}{dx} + y \right) = 0 \]

The first ODE is \( \frac{dy}{dx} = 0 \) or \( y = C \) so \( \Phi = y \), the second ODE is \( \frac{dy}{y} = -\frac{dx}{x} \) or \( y = C/x \) or \( xy = C \) so \( \Psi = xy \).

The method of the textbook does not correctly handle the next part of the problem. The method of textbook does work if \( A, B, C \) are constants. The additional work needed to solve this in this version of extra.

Step 3: The table in the text implies \( u = f(y) + g(xy) \) should be the solution. But it is not; checking we see that
\[ u_x = yg'(xy); \quad u_{xx} = y^2g''(xy); \quad u_{xy} = xyg''(xy) + g'(xy) \]
\[ xu_{xx} - yu_{xy} = xy^2g''(xy) - xy^2g''(xy) - yg'(xy) \neq 0 \]

Instead we need another trick.

The trick is to let \( p(x,y) = u_x \), our PDE becomes \( xp_x - yp_y \) which is a first order equation and which has the general solution \( p = g(xy) \) found above. (This is easy to check.) Now we just solve \( u_x = g(xy) \) by integration obtaining
\[ u = f(y) + \int g(xy) \, dx = f(y) + h(xy)/y \]

Why is the \( \int g(xy) \, dx = h(xy)/y \)? Well it has to be something whose \( x \)-partial is a function of \( xy \). So in must be an arbitrary function \( h(xy) \) but we need to make its \( x \)-partial, \( yh(xy) \), be a function of \( xy \); clearly dividing by \( y \) does the trick. Checking this solution gives
\[ u_x = yh'(xy)/y; \quad u_{xx} = yh''(xy); \quad u_{xy} = xh''(xy) \]
\[ xu_{xx} - yu_{xy} = xyh''(xy) - xyh''(xy) = 0 \]

Step 4: \( \xi = y, \eta = xy \) \( u_x = 0u_\xi + yu_\eta, u_y = u_\xi + xu_\eta, u_{xx} = y(0u_\eta + yu_\eta) = y^2u_\eta, u_{xy} = u_\eta + y(xu_\eta + u_\eta) = yu_\eta + xyu_\xi + u_\eta, u_{xx} = u_\xi + xu_\eta + x(u_\xi + xu_\eta) = u_\xi + 2xu_\xi + x^2u_\eta \)
\[ xu_{xx} - yu_{xy} = xy^2u_\eta - (y^2u_\eta + xy^2u_\xi + yu_\eta = xy^2u_\eta + yu_\eta \]

Dividing by \( xy^2 = y\eta \) we get the canonical
\[ u_\eta + u_\eta/\eta = 0 \]

since the second term is lower order we are ok.