Several Views of Integration by Parts

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1 Classic view

The classic integration by parts formula is

$$\int u \, dv = uv - \int v \, du$$

which comes from integrating the product rule \((fg)' = f'g + fg'\). When trying to use this formula on real integrals, there are a number of shortcuts and/or organized techniques that might make actual integration easier. Our example integral is

$$\int x^2 \cos nx \, dx$$

which should illustrate the different methods.

2 Step by step

First we blindly do parts and parts. First \(u = x^2\) and hence \(dv = \cos nx \, dx\); the next step is to find \(du = 2x \, dx\) and \(v = \frac{\sin nx}{n}\). The application of parts gives

$$\int x^2 \cos nx \, dx = x^2 \frac{\sin nx}{n} - \int 2x \frac{\sin nx}{n} \, dx$$

and we do the second application with \(u = 2x\) and \(dv = \frac{\sin nx}{n} \, dx\); again \(du = 2 \, dx\) and \(v = -\frac{\cos nx}{n^2}\). The second application of parts gives

$$\int x^2 \cos nx \, dx = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - \int 2 \frac{\cos nx}{n^2} \, dx$$

We can integrate this last integral to obtain:

$$\int x^2 \cos nx \, dx = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} + C$$

(Alternately one can do parts a third time.)

3 Guess and check

This method is basically guess and then correct. First we guess

$$F(x) = x^2 \frac{\sin nx}{n}$$

because the derivative of the factor containing \(\sin\) will match the integrand. But the product rule gives another term.

$$F'(x) = x^2 \cos nx + 2x \frac{\sin nx}{n}$$
We need to correct \( F(x) \) to cancel this term, guess
\[
F(x) = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2}
\]
and now
\[
F'(x) = x^2 \cos nx + 2x \frac{\sin nx}{n} - 2x \frac{\sin nx}{n} + 2 \frac{\cos nx}{n^2}
\]
We have a new term to cancel guess
\[
F(x) = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3}
\]
and finally
\[
F'(x) = x^2 \cos nx + 2 \frac{\cos nx}{n^2} - 2 \frac{\cos nx}{n^2}
\]
as required. Again
\[
\int x^2 \cos nx \, dx = x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} + 2 \frac{\sin nx}{n^3} + C
\]
Actually all Calculus 2 integration by parts problems are doable by guess and check. See the byguessnby-golly.pdf file.

4 Undetermined coefficients

Here we think of looking for a particular solution to the inhomogeneous problem \( y' = x^2 \cos nx \). Undetermined coefficients say to guess
\[
y = Ax^2 \sin nx + Bx^2 \cos nx + Cx \sin nx + Dx \cos nx + E \sin nx + F \cos nx
\]
We plug into the ODE
\[
y' = Anx^2 \cos nx + 2Ax \sin nx + Bnx^2 \sin nx + 2Bx \cos nx + Cnx \cos nx + C \sin nx + Dx \cos nx + E \cos nx + En \cos nx + Fn \sin nx
\]
Collecting like terms:
\[
y' = Anx^2 \cos nx - Bnx^2 \sin nx + (2A - Dn)x \sin nx + (2B + Cn)x \cos nx + (C - Fn) \sin nx + (D + En) \cos nx
\]
and we get the system of equations
\[
\begin{align*}
1 &= An & \text{\( x^2 \cos nx \) terms} \\
0 &= -Bn & \text{\( x^2 \sin nx \) terms} \\
0 &= 2B + Cn & \text{\( x \cos nx \) terms} \\
0 &= 2A - Dn & \text{\( x \sin nx \) terms} \\
0 &= D + En & \text{\( \cos nx \) terms} \\
0 &= C - Fn & \text{\( \sin nx \) terms}
\end{align*}
\]
So \( A = 1/n, B = 0, C = 0, D = 2/n^2, E = -2/n^3, F = 0 \). and we get the particular solution:
\[
y = \frac{1}{n} x^2 \sin nx + \frac{2}{n^2} x \cos nx - \frac{2}{n^3} \sin nx
\]
The general solution needs +C, which adds the general homogeneous solution. While not needed, a clever person could have figured out that \( B = C = F = 0 \) at the beginning.
5 Generalized parts

Let's rewrite parts as

\[ \int fg' = fg - \int f'g \]

and now apply it twice to

\[ \int fg'' = fg' - \int f'g' = fg' - (f'g - \int f''g) = f(g' - f'g + \int f''g) \]

And third time

\[ \int fg''' = fg'' - f'g' + f''' - \int f'''g \]

Applying to \( f = x^2 \) and \( g''' = \cos nx \) gives \( f' = 2x, f'' = 2, f''' = 0 \) and \( g'' = \sin nx/n, g' = -\cos nx/n^2 \) and \( g = -\sin nx/n^3 \) so plugging into the formula

\[ \int fg''' = x^2 \sin nx/n + 2x \cos nx/n^2 - 2 \sin nx/n^3 \]

The formulas continue.

\[ \int fg'''' = fg''' - f'g'' + f'''g - f'''' + \int f'''''g \]

6 Polynomial times \( g^{(n)}(x) \)

We apply the generalized formula's to the special case of a polynomial times a function we can integrate repeatedly. Our polynomial is \( x^2 \) our \( g^{(n)}(x) \) is \( \cos nx \) we form a table. The \( u \) column does derivative as we go down while the \( dv \) column does anti-derivatives. The third column is a diagonal product with alternating signs. The first entry in the third column comes from the diagonal \( x^2 \) to \( \sin nx/n \) and ends at \((+1)(x^2)(\sin nx/n)\). The next entry is the diagonal \( 2x \) to \(-\cos nx/n^2 \) to \((-1)(2x)(-\cos nx/n^2)\). The last entry is the diagonal \( 2 \) to \(-\sin nx/n^3 \) to \((+1)(2)(-\sin nx/n^3)\). Note the alternating signs \((+1), (-1), (+1), \) etc.

<table>
<thead>
<tr>
<th>( u )</th>
<th>( dv )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( \cos nx )</td>
</tr>
<tr>
<td>( 2x )</td>
<td>( \sin nx/n )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>(-\cos nx/n^2 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>(-\sin nx/n^3 )</td>
</tr>
</tbody>
</table>

The answer is

\[ \frac{x^2 \sin nx}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} + C \]