1 The technique is so nice, let’s do it twice

Some integrals can be done but using parts twice and then noticing the “leftover” integral is a scalar multiple of the original integral. This too can be done from our table. First we repeat the next section from the earlier text. Our example integrals are \( \int e^{2x} \sin 3x \, dx \) and \( \int \cos \pi x \cos 2n\pi x \).

2 Generalized parts

Let’s rewrite parts as

\[ \int fg' = fg - \int f'g \]

and now apply it twice to

\[ \int fg'' = fg' - \int f'g' = fg' - (f'g - \int f''g) = fg' - f'g + \int f''g \]

And third time

\[ \int fg''' = fg'' - f'g'' + f''g - \int f'''g \]

The formulas continue.

\[ \int fg'''' = fg'''' - f'g'''' + f''g'' - f'''g'' + \int f'''''g \]

3 Table revisited

We form a tables to express the above formula’s. As before, The \( u \) column does derivative as we go down while the \( dv \) column does anti-derivatives. As before we have the diagonal products, and now the leftover integral is the bottom row’s horizontal product. Both the diagonal’s and the horizontal products alternate signs. Here are the first 3 cases.
\[ \int fg''' = fg'' - f'g' + f''g - \int f'''g \]

\[ u \quad dv \]
\[ f' + g''' \]
\[ f'' - g'' \]
\[ f'' - g' \]
\[ f''' - g \]

4 Example A \( \int e^{2x} \sin 3x \, dx \)

We form a table to evaluate \( I = \int e^{2x} \sin 3x \, dx \)

\[
\begin{array}{c|c}
 u & dv \\
 \hline
 e^{2x} & \sin 3x \\
 2e^{2x} & -\cos 3x \\
 4e^{2x} & -\frac{3}{9} \\
\end{array}
\]

Note the bottom row is \(-\frac{4}{9}\) times the first.

\[
I = -e^{2x} \frac{\cos 3x}{3} + 2e^{2x} \frac{\sin 3x}{9} - \frac{4}{9}I
\]

\[
\frac{13}{9}I = -\frac{1}{3}e^{2x} \cos 3x + \frac{2}{9}e^{2x} \sin 3x
\]

\[
I = -\frac{3}{13}e^{2x} \cos 3x + \frac{2}{13}e^{2x} \sin 3x
\]

We need a constant of integration at the end.

5 Example B \( \int \cos \pi x \cos 2n\pi x \, dx \)

Again a table to evaluate \( I = \int \cos \pi x \cos 2n\pi x \, dx \)

\[
\begin{array}{c|c}
 u & dv \\
 \hline
 \cos \pi x & \cos 2n\pi x \\
 -\pi \sin \pi x & \sin 2n\pi x \\
 -\pi^2 \cos \pi x & -\frac{2n\pi}{(2n\pi)^2} \\
\end{array}
\]

Note the bottom row is \(\frac{1}{4n^2}\) times the first.

\[
I = \cos \pi x \frac{\sin 2n\pi x}{2n\pi} - \pi \sin \pi x \frac{\cos 2n\pi x}{(2n\pi)^2} + \frac{1}{4n^2}I
\]

\[
(1 - \frac{1}{4n^2})I = \frac{1}{2n\pi} \cos \pi x \sin 2n\pi x - \frac{1}{4n^2\pi} \sin \pi x \cos 2n\pi x
\]

\[
I = \frac{1}{(4n^2 - 1)\pi} (2n \cos \pi x \sin 2n\pi x - \sin \pi x \cos 2n\pi x)
\]

We need a constant of integration at the end.