Indeterminate Forms

$$\begin{align*}
0 & \cdot \infty, \quad 0 \cdot \infty, \quad 0^0, \quad \infty^0, \quad 1^\infty, \quad \infty - \infty
\end{align*}$$

These are the so called indeterminate forms. One can apply L'Hopital's rule directly to the forms $0/\infty$ and $\infty/\infty$. It is simple to translate $0 \cdot \infty$ into $0/\infty$ or into $\infty/0$, for example one can write $\lim_{x \to \infty} x e^{-x}$ as $\lim_{x \to \infty} x/e^x$ or as $\lim_{x \to \infty} e^{-x}/(1/x)$. To see that the exponent forms are indeterminate note that

$$\ln 0^0 = 0 \ln 0 = 0(-\infty) = 0 \cdot \infty, \quad \ln \infty^0 = 0 \ln \infty = 0 \cdot \infty, \quad \ln 1^\infty = \infty \ln 1 = \infty \cdot 0 = 0 \cdot \infty$$

These formula's also suggest ways to compute these limits using L'Hopital's rule. Basically we use two things, that $e^x$ and $\ln x$ are inverse functions of each other, and that they are continuous functions. If $g(x)$ is a continuous function then $g(\lim_{x \to a} f(x)) = \lim_{x \to a} g(f(x))$.

For example let's figure out $\lim_{x \to \infty} (1 + \frac{1}{x})^x$. This is of the indeterminate form $1^\infty$. We write $\exp(x)$ for $e^x$ so to reduce the amount exponents.

$$\lim_{x \to \infty} (1 + \frac{1}{x})^x = \exp(\ln(\lim_{x \to \infty} (1 + \frac{1}{x})^x)) = \exp(\lim_{x \to \infty} x \ln(1 + \frac{1}{x})) = \exp(\lim_{x \to \infty} x \ln(1 + \frac{1}{x})) / \lim_{x \to \infty} x$$

We can now apply L'Hopital's since the limit is of the form $0/0$.

$$= \exp(\lim_{x \to \infty} (1/\sqrt[3]{x})(-1/x^2)) = \exp(\lim_{x \to \infty} 1/(1 + \frac{1}{x})) = \exp(1) = e.$$ 

**Exercises I.** Find the limits.

A. $\lim_{x \to \infty} (1 + \frac{1}{x})^{3x}$  B. $\lim_{x \to \infty} (1 + \frac{k}{x})^x$  C. $\lim_{x \to 0} (1 + x)^{1/x}$  D. $\lim_{x \to 0} x^x$  E. $\lim_{x \to 0} x/(x^3)$  F. $\lim_{x \to 0} x/\ln x$.

One might be tempted to handle $\infty - \infty$ in a similar manner since

$$\frac{e^{\infty - \infty}}{e^{\infty + \infty}} = \frac{\infty}{\infty} = \frac{\infty}{\infty} = 0.$$ 

But L'Hopital's rule doesn't help here as the derivatives don't simplify. Instead, let $f(x)$ and $g(x)$ be functions so that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \infty$ so that $\lim_{x \to a} (f(x) - g(x)) = \infty - \infty$. One can rewrite $f(x) - g(x)$ as $f(x)(1 - g(x)/f(x))$. The limit $\lim_{x \to a} g(x)/f(x)$ is of the form $0/0$ and so we can use L'Hopital. If $\lim_{x \to a} (1 - g(x)/f(x)) = c \neq 0$ then $\lim_{x \to a} f(x)(1 - g(x)/f(x)) = c \lim_{x \to a} f(x)$ = sign of $c$. On the other hand, if $c = 0$, then $f(x)(1-g(x)/f(x))$ is of the form $0/0$ which we already know how to reexpress so that L'Hopital's rule can be applied.

For example let's show that $\lim_{x \to \infty}(\sqrt{x + 1} - \sqrt{x}) = 0$. This is of the indeterminate form $\infty - \infty$.

$$\lim_{x \to \infty}(\sqrt{x + 1} - \sqrt{x}) = \lim_{x \to \infty}(\sqrt{\frac{x + 1}{x}} - 1) = \lim_{x \to \infty}\sqrt{\frac{1}{x}}(\sqrt{1 + \frac{1}{x}} - 1) = \lim_{x \to \infty}(\frac{\sqrt{1 + \frac{1}{x}} - 1}{x^{-1/2}})$$

Now we can use L'Hopital's rule.

$$= \lim_{x \to \infty}(\frac{-1/x^2}{2x^{3/2}}) = \lim_{x \to \infty}(\frac{2x^{3/2}/x^2}{2\sqrt{1 + 1/x}}) = \lim_{x \to \infty}(\frac{1}{\sqrt{x}\sqrt{1 + 1/x}}) = 0.$$ 

**Exercises II.** Find the limits.

W. $\lim_{x \to \infty} ((x + 1)^3 - x^3)$  X. $\lim_{x \to \infty} (\ln(x + 2) - \ln(x))$  Y. $\lim_{x \to \infty} (3^x - 2^x)$  Z. $\lim_{x \to \infty} (x^2 - x^{-1})$