Indeterminate Forms

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad 0^0,\quad \infty^0,\quad 1^\infty,\quad \infty-\infty$$

These are the so called indeterminate forms. One can apply L'Hopital's rule directly to the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. It is simple to translate $0\cdot\infty$ into $\frac{0}{1/\infty}$ or into $\frac{\infty}{1/0}$, for example one can write $\lim_{x\to\infty} xe^{-x}$ as $\lim_{x\to\infty} x/e^x$ or as $\lim_{x\to\infty} e^{-x}/(1/x)$. To see that the exponent forms are indeterminate note that

$$\ln 0^0 = 0\ln 0 = 0(-\infty) = 0\cdot\infty,\quad \ln\infty^0 = 0\ln\infty = 0\cdot\infty,\quad \ln 1^\infty = \infty\ln 1 = \infty\cdot 0 = 0\cdot\infty$$

These formula's also suggest ways to compute these limits using L'Hopital's rule. Basically we use two things, that e^x and $\ln x$ are inverse functions of each other, and that they are continuous functions. If g(x) is a continuous function then $g(\lim_{x\to a} f(x)) = \lim_{x\to a} g(f(x))$.

For example let's figure out $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$. This is of the indeterminate form 1^∞. We write $\exp(x)$ for e^x so to reduce the amount exponents.

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = \exp(\ln(\lim_{x\to\infty}(1+\frac{1}{x})^x)) = \exp(\lim_{x\to\infty}\ln((1+\frac{1}{x})^x))$$
$$= \exp(\lim_{x\to\infty} x\ln(1+\frac{1}{x})) = \exp(\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{1/x})$$

We can now apply L'Hopital's since the limit is of the form $\frac{0}{0}$.

$$= \exp(\lim_{x\to\infty}\frac{(1/(1+\frac{1}{x}))(-1/x^2)}{-1/x^2}) = \exp(\lim_{x\to\infty} 1/(1+\frac{1}{x})) = \exp(1) = e.$$

Exercises I. Find the limits.

$$A.\ \lim_{x\to\infty}(1+\frac{1}{x})^{3x}\quad B.\ \lim_{x\to\infty}(1+\frac{k}{x})^x\quad C.\ \lim_{x\to 0}(1+x)^{1/x}\quad D.\ \lim_{x\to 0^+} x^x\quad E.\ \lim_{x\to 0^+} x^{(x^2)}\quad F.\ \lim_{x\to 0^+} x^{1/\ln x}.$$

One might be tempted to handle $\infty-\infty$ in a similar manner since

$$e^{\infty-\infty} = \frac{e^\infty}{e^\infty} = \frac{\infty}{\infty}.$$

But L'Hopital's rule doesn't help here as the derivatives don't simplify. Instead, let $f(x)$ and $g(x)$ be functions so that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ so that $\lim_{x\to a}(f(x)-g(x))$ is $\infty-\infty$. One can rewrite $f(x)-g(x)$ as $f(x)(1-g(x)/f(x))$. The limit $\lim_{x\to a} g(x)/f(x)$ is of the form $\frac{\infty}{\infty}$ and so we can use L'Hopital. If $\lim_{x\to a}(1-g(x)/f(x)) = c \neq 0$ then $\lim_{x\to a} f(x)(1-g(x)/f(x)) = c\lim_{x\to a} f(x)$ =sign of $c\infty$ (sign of c is $\pm$ depending on the sign of c. On the otherhand, if $c=0$, then f(x)(1-g(x)/f(x)) is of the form $0\cdot\infty$ which we already know how to reexpress so that Hopital's rule can be applied.

For example let's show that $lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x}) = 0$. This is of the indeterminate form $\infty-\infty$.

$$lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x}) = lim_{x\to\infty}\sqrt{x}(\frac{\sqrt{x+1}}{\sqrt{x}}-1) = lim_{x\to\infty}\sqrt{x}(\sqrt{1+\frac{1}{x}}-1) = lim_{x\to\infty}\frac{(\sqrt{1+\frac{1}{x}}-1)}{x^{-1/2}}$$

Now we can use L'Hopital's rule.

$$= lim_{x\to\infty}\frac{\frac{-1/x^{-2}}{2\sqrt{1+1/x}}}{(-1/2)x^{-3/2}} = lim_{x\to\infty}\frac{2x^{3/2}/x^{-2}}{2\sqrt{1+1/x}} = lim_{x\to\infty}\frac{1}{\sqrt{x}\sqrt{1+1/x}} = 0.$$

Exercises II. Find the limits.

$$W.\ \lim_{x\to\infty}((x+1)^3-x^3)\quad X.\ \lim_{x\to\infty}(\ln(x+2)-\ln(x))\quad Y.\ \lim_{x\to\infty}(3^x-2^x)\quad Z.\ \lim_{x\to 0}(x^{-2}-x^{-1})$$