Lines in Space

This short section is about equations of lines in three dimensions. The first equation is analogous to the point-slope form in two dimensions. The second is analogous to the the two point form. The equation

$$\mathbf{X} = \mathbf{X_0} + \mathbf{D}t \tag{1v}$$

is the vector equation of the line going throught the "point" or rather the vector $\mathbf{X_0} = \langle x_0, y_0, z_0 \rangle = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ and in the direction $\mathbf{D} = \langle a, b, c \rangle$. Equivalently, the vector equation above is equivalent to the three scalar equations

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct \tag{1s}$$

These are sometimes called parameteric equations, since the parameter t need not be one of the coordinate axes. The equation of the line through the vectors $\mathbf{X_0}$ and $\mathbf{X_1}$ can be obtained from equation 1 by letting $\mathbf{D} = \mathbf{X_1} - \mathbf{X_0}$.

Note when $t = 0$ the curve is at the vector $\mathbf{X_0}$ or the point (x_0, y_0, z_0) and in the second case the curve is at $\mathbf{X_1}$ when $t = 1$. Some example problems. Maple will plot lines, and more general parametric equations with the spacecurve command (needs with(plots);), for example the command below will plot the parameteric equations $x = \sin t, y = \cos t, z = 2\pi t$ which is the graph of a helix.

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▷ spacecurve([sin(t),cos(t),2*Pi*t],t = 0..3, title='3 turns of a Helix');
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#1 Find the vector equation of the line through the points $(1, 0, 1)$ and $(3, 2, 0)$.

Ans. $\mathbf{D} = \langle 3-1, 2-0, 0-1 \rangle = \langle 2, 2, -1 \rangle$. So $\mathbf{X} = \langle 1, 0, 1 \rangle + \mathbf{D}t = \langle 1 + 2t, 2t, 1 - t \rangle$.

#2 Find where the line $x = 1 + t, y = t, z = 1 - t$ intersects the sphere $x^2 + y^2 + z^2 = 14$.

Ans. Substituting in we have

$$\begin{aligned} (1+t)^2 + t^2 + (1-t)^2 &= 14 \\ 1 + 2t + t^2 + t^2 + 1 - 2t + t^2 &= 14 \\ 2 + 3t^2 &= 14 \\ t^2 &= 4 \\ t &= -2 \quad or \quad 2 \end{aligned}$$

So when $t = 2$ we have the point of intersection $(3, 2, -1)$ and when $t = -2$ we have the point of intersection $(-1, -2, 3)$

#3 Find where the two lines $x = 1 + t, y = 2 + t, z = 3 + t$ and $x = 1 + 3s, y = 1 + 4s, z = 1 + 5s$ intersect.

Ans. Note that not every pair of lines intersect. Parallel lines in the plane to do not intersect, in space there are non-parallel lines (skew lines) which do not intersect. The solution is to solve the equations the three equations $1 + t = 1 + 3s, 2 + t = 1 + 4s, 3 + t = 1 + 5s$ in two unknowns. If it has no solution the lines do not intersect, if it has infinitely many solutions, the lines are the same, and if it has a unique solution it will give the point of intersection. In practice, if is often fastest to solve two of equations say $1+t = 1+3s, 2+t = 1+4s$ and substitute in the third to see if it is a solution to the third as well. Here if we substract equation 1 from 2 we get $1 = s$ and so $1 + t = 1 + 3(1)$ or $t = 3$. In this case $t = 3, s = 1$ is a solution to the third equation as well so $(1+3, 2+3, 3+3) = (4, 5, 6) = (1+3(1), 1+4(1), 1+5(1))$ is the point of intersection. Note if $z = 1 + 5s$ is replaced by $z = 0 + 5s$, then the lines do not intersect.

#4 Find the equation of a line through $(1, 1, 1)$ parallel to $\mathbf{X} = \langle 3 - t, 2 - 2t, 5t \rangle$.

Ans. The lines having the same direction are parallel. So we can use the direction $\mathbf{D} = \langle -1, -2, 5 \rangle$, so $\mathbf{X} = \langle 1, 1, 1 \rangle + \langle -1, -2, 5 \rangle\, t$ is a solution.

#5 Find the equation of line which is the intersection of the two planes $x + y + z = 3$ and $x - y + z = 5$

Ans . Solve for two variables in terms of the third. Solving $x + y = 3 - z, x - y = 5 - z$ yields $x = 4 - z, y = -1$ which yield the parametric equations $x = 4 - z, y = -1, z = z$ or perhaps the equivalent form $x = 4 - t, y = -1 + 0t, z = 0 + t$ looks more satisfying.

#6 Find the equation of the y-axis.

Ans. A point on the line is $(0, 0, 0)$ and the direction is $\mathbf{j} = \langle 0, 1, 0 \rangle$ which yields the equations $x = 0, y = t, z = 0$

Exercise: Find the coordinates of the point on the plane $x + 2y + 4z = 4$ nearest the origin. Repeat for the plane $Ax + By + Cz = D$.