Orbit 13

Yes those famous words, “Class, we have a problem.” once again orbit the air waves. This time we are representing the famous special effects firm of Industrial Light and Mathematics who are considering doing the special effects for the up coming block-buster film “Orbit 13”. Major actors like Tom Hunks and Kathleen Quizon are set to star in this Sci-Fi action drama directed by Run Howbackward. Run wants I.L. & M. to do the actual orbits calculation and some preliminary graphics.

Mr Howbackward (on flimsy Hollywood grounds) has chosen to ignore perhaps the greatest scientific discovery of all time and to have all the orbits based on perfect circles rather than the ellipses that Kepler discovered. Run tells you that he had an intern from FSU who has already explained to him that planets rotate counter-clockwise about a star when you look from above the star’s north pole and the same is true for moons orbiting planets. Furthermore, tradition puts the Sun at the origin with its north pole pointed up. The intern added a hypothetical Earth-Moon system where the axis of the Earth was parallel to the axis of the Sun and not tilted 24 degrees like our Earth. Thus the intern was able to put all three bodies, the Sun, the Earth and the Moon in the xy-plane.

The intern’s equation for the Moon in this system was based on having the Earth a radius R from the Sun, the Moon a radius of 1 from the Earth and having the Moon orbit the Earth 12 times for every one time the Earth orbits the sun. The equation is below.

$$\mathbf{M}(t) = \langle R\cos(t) + \cos(12t), R\sin(t) + \sin(12t)\rangle$$

Unfortunately for Mr Howbackward, the intern graduated and left. So your first job:

a. Explain why $\mathbf{M}(t)$ is the parametric equation of the moon.

Run tells you that one of the plot twists puts a requirement on the system. The Moon must stop for an instant relative to the Sun so that a rescue can occur. It is time to do some initial graphics too.

b. Find R and t such that the moon stops relative to Sun at time t.

c. Prepare three (different) plots of the orbit of moon. The first uses the value of R you found in part (b). The second uses $R/2$ and third $2R$.

Looking over your shoulder at your wonderful graphs, Tom Hunks says these orbits don’t look convex. “Can’t you make the orbit convex?” He asks. You explain that the lack of convexity is caused by the orbit having a radius of curvature whose center is outside the orbit. You think for a second as you see the lack of comprehension on everyones face and realize that radius of curvature is no longer covered in Calculus 3 texts. But being fast on your feet you come up with a quick approximation. The orbit will always be convex, if the moon’s acceleration is always pointed inward. Tom suddenly understands and says “you are saying that if the scalar projection of $\mathbf{M}''(t)$ onto $\mathbf{M}(t)$ is never positive then the orbit will be convex. Lets find the smallest such R and plot it.”

d. Find the smallest positive R so that this scalar projection is less than or equal to zero for all t and plot the orbit for this R.

Another actor, Kevin Broccoli, is looking over the orbits you drew in part (c) and counts the points. Kevin comments ”There are not twelve of them. If the Moon goes around the Earth 12 times for Earth orbit what do these points represent?” Looking at Kevin, Kathleen Quizon says “Those represent the new moon phase which happens when the Moon is directly between the Earth and the Sun. The full moon phase happens between two new moons and it occurs when the Earth is directly between the Sun and Moon. Kevin, it is the difference between sidereal and synodic months.”

e. Explain why the number of full moons is not 12 and why synodic months are longer than sidereal months. Modify the equation of the Moon so there are exactly 12 full moons for each orbit of the Earth and plot it for the new R that satisfies the condition in part (b) for the modified orbit and plot the result.

There is a lull while everyone is enjoying looking at these orbits. You smile and say, “These are only two dimensional plots, we can get three dimensional orbits if we model our system on some moon of Uranus.”

Run thinks aloud "Will a film using the word 'Uranus' be taken seriously?" You reply, that it is also the only planet whose moons are named for characters in English literature, there is a moon named Juliet for instance. But you continue, pointing out that Uranus almost has its axis in the same plane as its orbit. Currently its north pole is pointing away from the Sun and we could approximate this by having the north pole always pointing in the **i**-direction.

f. Let the Sun-Uranus distance be R, the Uranus-Juliet distance be r and suppose Juliet rotates N times Uranus (sidereal) for every rotation of Uranus about the Sun. Remember the north pole of Uranus is in the **i**-direction so looking in negative x-direction you should see Juliet orbiting counter-clockwise. Derive the parametric equation for Juliet in this system. Do a 3D tubeplot of this orbit using the parameters $N = 6, r = 1$ and $R = 2$.

Run thinks for a second and says, "does the axis have to always point in the same direction?" Run pushes aside your attempt to explain angular momentum, and says "What if the north pole of Uranus is always pointed in the direction of Uranus' velocity. Then the Juliet orbit would always be on the surface of a torus. Can we look at that?"

g. Derive parametric equations for this toral hugging orbit of Juliet. Do a 3D tubeplot of this orbit using the parameters $N = 6, r = 1$ and $R = 2$.

Kevin points out that since the moon always keeps one face towards the earth maybe the movie going public would buy Uranus always having the south pole pointed directly at the Sun at all times. Yes it is a third orbit for Juliet, but it is the last.

h. Derive parametric equations for this south is toward the sun orbit of Juliet. Do a 3D tubeplot of this orbit using the parameters $N = 6, r = 1$ and $R = 2$.

Your job is done, but curiosity is too much and you ask Mr Howbackward why no ellipses. He looks around and sees that no one is listening and replies "Our biggest product placement customer's main competitor has an ellipse for a logo."

From the Course Syllabus

PROJECT. You will work on the project in groups of 1–4 students. This project will be a substantial assignment, giving you a chance to earn part of your grade in an environment which simulates the so-called "real world" better than does an in-class exam. It will also give your instructor a chance to base part of your grade on your best work, produced in a setting where time should not be a factor (assuming you start on your project as soon as it is assigned). The results of your work on your project will be presented in a report (one report per group). Each member will also submit a "group evaluation" giving their impression of the relative contribution of each member to the group's effort. These evaluations are due with the project. It is not guaranteed that each member of the group will receive the same grade. The reports will be graded not only on their mathematical content but also on the quality of the presentation: clarity, neatness, and proper grammar are also important. **Both reports and group evaluations must be typed.** The project will be assigned on Thursday, March 7 and due on Thursday, March 28.

Grading

Each part is worth 10 points which leaves 20 points for clarity, neatness, grammar and general wow value, for a total of 100 points. You are free to use Maple on all the calculations, but for clarity you should do the easy ones by hand. There are a very small number of bonus points.

If you like the statement of this project, you might also like:
http://movieweb.com/movie/apollo13/ or http://us.imdb.com/Title?0112384
Also some maple drawn graphics illustrating some of objects in the project is available at:
http://www.math.fsu.edu/~bellenot/class/s02/cal3/project.html