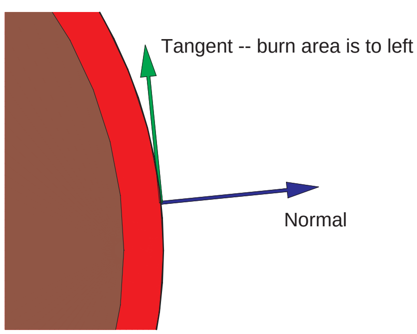
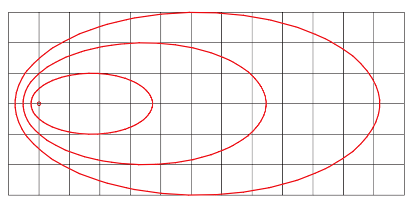
Wildfire – or Smokey wants you!

The National Prairie Park Service has hired your firm "Math Afire" to help it understand prairie fires. Your firm is to model the fire perimeter. Fires can be complex, but as a first approximation the prairie can be modeled as a flat plane and the plane can be divided into 3 regions. The 3 regions are: the large unburned area outside the fire, the fire perimeter where the flames are actively burning, and the burnt area where the fire has passed. Generally speaking the fire line can be thought of as being composed of many adjacent short thin arcs, and on each arc the fire grows or moves normal to this arc. The fire can spread at different rates on the different arcs.

Part 1. Parametric Fire Lines.

Group P (P for parametric) decides to model the boundary of the fire using a parametric equation $\mathbf{r}(s) = \langle x(s), y(s) \rangle$. Mentally they freeze time and march around the fire line using s as parameter. This gives the fire line as curve at some time t. Fire fighters can approximate such a curve by walking around the fire with a GPS. The curve parameterization is chosen so that as s increases the fire is always to the left. This means the curve goes around a circular or elliptical fire in the counterclockwise direction. Notice that the parameter is s and not t. As the fire burns the parametric curve needs to move too. Thus we can think of the parametric curve as being a function of two variables $\mathbf{r}(s,t) = \langle x(s,t), y(s,t) \rangle$. Where $\mathbf{r}(s,0) = \langle x(s,0), y(s,0) \rangle$. is the parametric equation for time $t = 0$ and $\mathbf{r}(s,1) = \langle x(s,1), y(s,1) \rangle$. is the parametric equation for time $t = 1$ and so on. It is convenient to think of the s parameter as not being arclength, which changes as t changes, but as a variable that always has the same range. So s could be percentage of total length (so $0 \le s \le 1$), or perhaps some angle (so $0 \le s \le 2\pi$).





So how does group P model the fire movement? (See figure to above left) If $\langle x(s), y(s) \rangle$. is the equation at time $t = 0$. Then the fire will be spreading to the right. So we need the "right" pointing unit normal. Show that $\mathbf{N}(s) = \langle y'(s), -x'(s) \rangle / \sqrt{x'(s)^2 + y'(s)^2}$ is this vector. *Checklist 1. unit length. 2. perpendicular to curve. 3. The angle from tangent to normal is always clockwise. [Hint cross product must have what kind (positive or negative) z component.]* So if the fire is spreading a $k(s)$ units per unit time at the point given by $\mathbf{r}(s)$, then the curve at time Δt later is given by $\langle x(s), y(s) \rangle + \Delta t k(s) \mathbf{N}(s)$. Show if the boundary is a circle of radius R at time $t = 0$ and k is constant, then the boundary will be a circle of radius $R + k\Delta t$ at time Δt later. *Checklist give parametric equations for $t = 0$, derive the equation for $t = \Delta t$, is it a circle?*

Part 2. Elliptic Fire Lines.

Simple lab experiments show that fires grow as ellipses (all other things being equal). The elliptical shape is caused usually by wind, but can also be caused by a slight slope in the terrain. Fires create their own wind and sometimes their own weather. The figure to the above right shows some experimental data. The fire started at the common focus of the ellipses given by the dot and spread mainly to the downwind direction to the right. But notice there is a smaller upwind velocity and an intermediate flank velocity at the points where the tangent lines are horizontal. Using the experimental data, estimate the downwind speed, the upwind speed and the flank speed of the fire at the time when the perimeter is at the middle ellipse. Be careful with the flank speed as the flank of one ellipse does not grow to the flank of the next ellipse. Show the path of the fire from the inner ellipse that goes through the horizontal tangent of the middle ellipse and then to outer ellipse. *Checklist 3 numerical estimates of speeds which include the coordinates of the points used and the flank path.* There is a bigger picture of the ellipses at the end of this assignment.

Group P found an absent minded professor who was ancient enough to know how to write equations of conic sections in polar coordinates with one focus at the origin. The professor claimed the the polar equation below was what they wanted, where e is the eccentricity of the conic and $p > 0$. (See the appendix on conic sections.)

$$r = \frac{2ep}{1 - e\cos\theta}$$

Group P estimated the eccentricity of the ellipse in our figure and decided that eccentricity was constant. *Checklist estimate the eccentricity of each ellipse.* So the parameter p controlled the size of the ellipse, as p increased so did the size of the ellipse.

Group P converted a polar equation like $r = r(\theta)$ to the parameter equation $\langle r(\theta)\cos\theta, r(\theta)\sin\theta\rangle$ in the parameter θ. But they were not so trusting of the old professor. Show that if $0 \le e < 1$, then the polar equation can be converted to rectangular coordinates and has the correct form for an ellipse with one focus at the origin. *checklist, a whole mess of algebra. Compute a, b, c and x_0 as functions of e and p.*

There are problems with the parametric model when 1. Two or more fires are merging into one fire. 2. When the fire-line is not convex. *Checklist why are these problems for the P model?* The rangers point out that **spotting** is a natural part of fires. Spotting is when embers carried by the wind start a new separate fire. And when these two fires combine, the perimeter is no longer convex.

Part 3. Level Set Fire Lines.

Group LS (LS for Level Set) has a model, which at the cost of an increase of one dimension, does not suffer from the defects of the parametric model above. The LS group decides to model the boundary of the fire as a contour of a more complex function $F(x, y)$. The boundary of the fire at time t is given by the contour (level curve) of $F(x, y) = t$. So the function $F(x, y) = t$ can be defined by the relation: the fire reaches the point (x, y) at the time t.

Redo the simple circle example. At $t = 0$ the radius of the circle is R and the fire is increasing a constant rate of k, find the equation of $F(x, y)$ and show its crosssection at Δt is a circle of radius $R + k\Delta t$. *Checklist one equation, two level curves, and a plot3d with style patchcontour.*

Redo the normal example, show that the unit normal at (x, y) is $\nabla F(x, y)/||\nabla F(x, y)||$ and the speed is $k(x, y) = 1/||\nabla F(x, y)||$. *Checklist 1. unit length. 2. perpendicular to level curve. 3. The normal is always pointed away from the fire. 4. correct speed.* Be careful, your $F(x, y)$ must be general and not a particular example.

Redo the elliptical examples, find $F(x, y)$ so that the level curves are ellipses with the same eccentricity and one focus at the origin. *Checklist one equation, one generic level curve, and a plot3d with style patchcontour.*

Combining fires, Show how three circular shaped fires increasing at at the same constant rate centered at (0,0), (2,1) and (0,2) will merge using the level set method. [Hint If $F00(x, y)$, $F21(x, y)$ and $F02(x, y)$ are the equations for the three fires separately then $F(x, y) = min(F00(x, y), F21(x, y), F02(x, y))$ is the equation for the three fires together. *Checklist one equation, several level curves showing the complete evolution from three separate fires to one almost convex fire, and a plot3d with style patchcontour.*

Appendix. Conic Sections

In rectangular coordinates the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $a > b > 0$ has semi-major diameter a and semi-minor diameter b. The curve is wider that it is tall. The ellipse goes through the points, $(\pm a, 0)$ and $(0, \pm b)$. Two important points for the ellipse are the foci located at $(\pm c, 0)$ where $c = \sqrt{a^2 - b^2}$. An important constant for the ellipse is its eccentricity e given by c/a. (This is not to be confused with Euler's $e = 2.718281828459045\ldots$). The eccentricity is a measure of the flatness of the ellipse. A circle has $e = 0$ and in general, $0 \le e < 1$ for an ellipse, flatter ellipses have larger eccentricities. [Aside: Parabolas have $e = 1$ and hyperbolas have $e > 1$.] To center the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the point (x_0, y_0) one changes the equation to

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1$$

From the course Syllabus

BIG PROJECT. You will work on the project in groups of 1–4 students. This project will be a substantial assignment, giving you a chance to earn part of your grade in an environment which simulates the so-called "real world" better than does an in-class exam. It will also give your instructor a chance to base part of your grade on your best work, produced in a setting where time should not be a factor (assuming you start on your project as soon as it is assigned). The results of your work on your project will be presented in a report (one report per group). Each member will also submit a "group evaluation" giving their impression of the relative contribution of each member to the group's effort. These evaluations are due with the project. It is not guaranteed that each member of the group will receive the same grade. The reports will be **graded not only on their mathematical content but also on the QUALITY of the presentation: CLARITY, NEATNESS, and PROPER GRAMMAR are also important**. Both reports and group evaluations must be **TYPED**. The project was be assigned on Thursday, March 20 and due on Thursday, April 3.

GRADING. The big project is worth 100 points or five times the value of an ordinary project but only twice the work time. Mathematical content is worth 80% which leaves 20 points for clarity, neatness, grammar and general wow value, for a total of 100 points. You are free to use Maple on all the calculations, but for clarity you should do the easy ones by hand. There are a very small number of bonus points.

The web page for the project, which at least has some amusing pictures, is located at this URL:
http://www.math.fsu.edu/~bellenot/class/s03/cal3/project.html