MAC 2313 Calculus 3 **Mini-Test 6** 14 Apr 2004

Directions: Show **ALL** work for credit; Give **EXACT** answers when possible; Start each problem on a **SEPARATE** page; Use only **ONE** side of each page; Be neat; Leave margins on the left and top for the **STAPLE**; Calculators can be used for graphing and calculating only; Nothing written on this page will be graded;

1. Compute div $\langle x\ln(x+y), x\ln(xy), \ln(x/z)\rangle$ and curl $\left\langle xyz, x^2y^2z^2, 0\right\rangle$

   Write $x\ln xy$ as $x\ln x + x\ln y$ and $\ln(x/z)$ as $\ln x - \ln z$, so the div becomes $\ln(x+y) + \frac{x}{x+y} + \frac{x}{y} - \frac{1}{z}$. The curl is $\left\langle -2x^2y^2z, xy, 2xy^2z^2 - xz\right\rangle$.

2. Let C be the line from $(0,1,0)$ to $(5,1,-1)$ and let $\vec{F} = \left\langle z, z^2+3y^2, x+2yz+2z\right\rangle$ compute the line integral $\int_C \vec{F}\cdot d\vec{r}$ two ways. First do it directly, and secondly do it using the fundamental theorem of line integrals.

   Directly $\vec{r} = \langle 5t, 1, -t\rangle$ for $0 \le t \le 1$. So $\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^1 \left\langle -t, t^2+3, 5t-2t-2t\right\rangle \cdot \langle 5, 0, -1\rangle\, dt = \int_0^1 -5t - t\, dt = -3$

   Indirectly, $f = xz + yz^2 + y^3 + z^2$ and $f(5,1,-1) - f(0,1,0) = (-5+1+1+1) - (1) = -3$

3. Are the following statements true or false?

   (a) The curl $\vec{F}$ is a scalar field. — False

   (b) The div $\vec{F}$ is a scalar field. — True

   (c) The $\nabla\vec{F}$ is a vector field. — False

   (d) A vector field $\vec{F}$ is a gradient vector field if its diverence is zero. — False

   (e) $\int_C \vec{F}\cdot d\vec{r} = \vec{F}(Q) - \vec{F}(P)$ where P and Q are the endpoints of C. — False

   (f) If $\vec{F}$ is path independent, then $\oint_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve C. — True

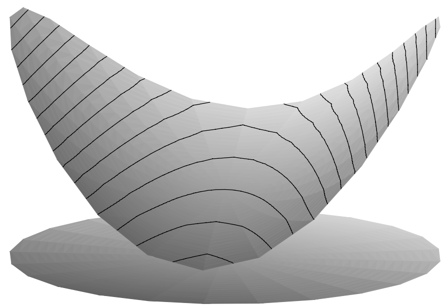
   (g) If $\vec{F} = \langle -y, x\rangle$ and C is a closed curve oriented counterclockwise which is ∂D for some region D in the plane, then $\oint_C \vec{F}\cdot d\vec{r} = 2\,\text{area}(D)$. — True

   (h) The vector field $\vec{F} = \vec{r}$ in 3-space is path independent. — True

   (i) If S_1 is a rectangle with area 1 and S_2 is a rectangle with area 2, then $2\iint_{S_1} \vec{F}\cdot d\vec{A} = \iint_{S_2} \vec{F}\cdot d\vec{A}$. — False

   (j) If $\vec{F} = 2\vec{G}$, then $\iint_S \vec{F}\cdot d\vec{A} = 2\iint_S \vec{G}\cdot d\vec{A}$. — True

4. Compute the flux integral of $\vec{F} = \langle x, y, z \rangle$ over the surface S (below) which is the portion of $z = f(x, y) = 10 + x^2 + y$ over the circle of radius 5 centered at the origin and is oriented upward.



We have $d\vec{A} = \langle -2x, -1, 1 \rangle$ $\vec{F}(S) = \big\langle x, y, 10 + x^2 + y \big\rangle$ so flux $= \int\int_{\text{disk}} -2x^2 - y + 10 + x^2 + y \, dA$ $= \int_0^{2\pi} \int_0^5 (10 - r^2 \cos^2 \theta) r \, dr \, d\theta = \int_0^{2\pi} 125 - 625 \cos^2 \theta / 4 \, d\theta = 250\pi - 625\pi/4 = 375\pi/4$

5. Check Gauss's Theorem by computing both integrals below. Let H be the the half of the ball of radius 2 where $z \geq 0$ and let $S = \partial H$ be the outward-oriented surface which is the union of a hemisphere T on top and a flat disk B on bottom. Use the vector field $\vec{F} = \langle 0, 0, 1 + z \rangle$.

$$\iiint_H \operatorname{div} \vec{F} \, dV = \iint_S \vec{F} \cdot d\vec{A}$$

LHS: div is 1 so the triple integral on the left is the volume of H or $\frac{2}{3}\pi 2^3 = 16\pi/3$. RHS: On B, the normal is $-\vec{k}$, and dot product of $\vec{F}$ with the normal is the constant -1. The area of the disk is $\pi 2^2$ so the flux is -4π. There are several ways to compute the flux over T, using spherical formula yields

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \langle 0, 0, 1 + 2\cos\phi \rangle \cdot \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle \, 2^2 \sin\phi \, d\phi \, d\theta$$

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 4\cos\phi \sin\phi + 8\cos^2\phi \sin\phi \, d\phi \, d\theta = 2\pi(-2\cos^2\phi - \frac{8}{3}\cos^3\phi)|_{\phi=0}^{\pi/2} = 28\pi/3$$

And $\frac{16}{3}\pi = -\frac{12}{3}\pi + \frac{28}{3}\pi$