Directions: Show ALL work for credit; Give EXACT answers when possible; Start each problem on a SEPARATE page; Use only ONE side of each page; Be neat; Leave margins on the left and top for the STAPLE; Calculators can be used for graphing and calculating only; Nothing written on this page will be graded.

1. Compute $\text{div} \langle x \ln(x+y), x \ln(xy), \ln(x/z) \rangle$ and curl $\langle xyz, x^2y^2z^2, 0 \rangle$

Write $x \ln xy$ as $x \ln x + x \ln y$ and $\ln(x/z)$ as $\ln x - \ln z$, so the div becomes $\ln(x+y) + \frac{x}{x+y} + \frac{x}{y} - \frac{1}{z}$.

The curl is $\langle -2x^2y^2z, xy, 2xy^2z^2 - xz \rangle$.

2. Let $C$ be the line from $(0,1,0)$ to $(5,1,-1)$ and let $\vec{F} = \langle z, z^2 + 3y^2, x + 2yz + 2z \rangle$ compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ two ways. First do it directly, and secondly do it using the fundamental theorem of line integrals.

Directly $\vec{r} = \langle 5t, 1, -t \rangle$ for $0 \leq t \leq 1$. So $\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle -t, t^2 + 3, 5t - 2t - 2t \rangle \cdot (5,0,-1) \, dt = \int_0^1 -5t - t \, dt = -3$.

Indirectly, $f = xz + yz^2 + y^3 + z^2$ and $f(5,1,-1) - f(0,1,0) = (-5 + 1 + 1 + 1) - (1) = -3$.

3. Are the following statements true or false?

(a) The curl $\vec{F}$ is a scalar field. — False
(b) The div $\vec{F}$ is a scalar field. — True
(c) The $\nabla \vec{F}$ is a vector field. — False
(d) A vector field $\vec{F}$ is a gradient vector field if its divergence is zero. — False
(e) $\int_C \vec{F} \cdot d\vec{r} = \vec{F}(Q) - \vec{F}(P)$ where $P$ and $Q$ are the endpoints of $C$. — False
(f) If $\vec{F}$ is path independent, then $\oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve $C$. — True
(g) If $\vec{F} = (-y, x)$ and $C$ is a closed curve oriented counterclockwise which is $\partial D$ for some region $D$ in the plane, then $\oint_C \vec{F} \cdot d\vec{r} = 2\text{area}(D)$. — True
(h) The vector field $\vec{F} = \vec{r}$ in 3-space is path independent. — True
(i) If $S_1$ is a rectangle with area 1 and $S_2$ is a rectangle with area 2, then $2\iint_{S_1} \vec{F} \cdot d\vec{A} = \iint_{S_2} \vec{F} \cdot d\vec{A}$. — False
(j) If $\vec{F} = 2\vec{G}$, then $\iint_S \vec{F} \cdot d\vec{A} = 2\iint_S \vec{G} \cdot d\vec{A}$. — True
4. Compute the flux integral of $\vec{F} = \langle x, y, z \rangle$ over the surface $S$ (below) which is the portion of $z = f(x, y) = 10 + x^2 + y$ over the circle of radius 5 centered at the origin and is oriented upward.

We have $d\vec{A} = \langle -2x, -1, 1 \rangle$ so flux $= \iint_{\text{disk}} -2x^2 - y + 10 + x^2 + y \, dA$ $= \int_0^{2\pi} \int_0^5 (10 - r^2 \cos^2 \theta) r \, dr \, d\theta = \frac{250\pi}{4} - \frac{625\pi}{4} = \frac{375\pi}{4}$

5. Check Gauss’s Theorem by computing both integrals below. Let $H$ be the half of the ball of radius 2 where $z \geq 0$ and let $S = \partial H$ be the outward-oriented surface which is the union of a hemisphere $T$ on top and a flat disk $B$ on bottom. Use the vector field $\vec{F} = \langle 0, 0, 1 + z \rangle$.

$LHS: \ \text{div is 1 so the triple integral on the left is the volume of } H \ \text{or } \frac{2}{3}\pi 2^3 = \frac{16\pi}{3}$. $RHS: \ \text{On } B, \ \text{the normal is } -\vec{k}, \text{ and dot product of } \vec{F} \ \text{with the normal is the constant } -1. \ \text{The area of the disk is } \pi 2^2 \ \text{so the flux is } -4\pi. \ \text{There are several ways to compute the flux over } T, \ \text{using spherical formula yields}$

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} (0, 0, 1 + 2 \cos \phi) \cdot (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \, 2\sin \phi \, d\phi \, d\theta$$

$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} 4 \cos \phi \sin \phi + 8 \cos^2 \phi \sin \phi \, d\phi \, d\theta = 2\pi (-2 \cos^2 \phi - \frac{8}{3} \cos^3 \phi)|^{\pi/2}_{\phi=0} = 28\pi/3$$

And $\frac{16\pi}{3} = -\frac{12\pi}{3} + \frac{28\pi}{3}$