Practice Mini-Test 3 – Calculus 3 – Spring 04

1. T2#1 F03 Chain Rule. [Compare T2#2 S03 T2#1 F02] Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$ when $z = \tan(x^2 - y^2)$, $x = s/t$ and $y = \sqrt{t}$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$= (2x\sec^2(x^2 - y^2))(\frac{1}{t}) + (-2y\sec^2(x^2 - y^2))(0)$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$= (2x\sec^2(x^2 - y^2))(\frac{-s}{t^2}) + (-2y\sec^2(x^2 - y^2))(\frac{1}{2\sqrt{t}})$$

2. T2#6 F03 Tangent Plane. [Compare T2#6 S03 T2#7 F02] Find the equation of the tangent plane to the level surface $F(x, y, z) = x^2 + 2y^2 + 6xy - 8x^3z + 27 = 0$ at $(1, -2, 3)$.

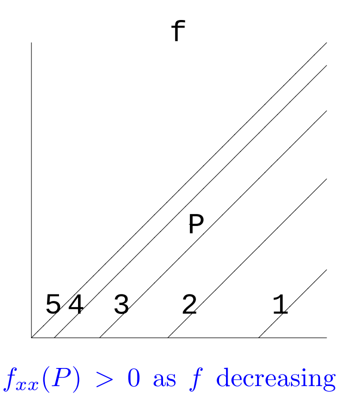
   $\nabla F = \left\langle 2x + 6y - 24x^2z, 4y + 6x, -8x^3 \right\rangle$ so $\nabla F(1, -2, 3) = \langle 2 - 12 - 72, -8 + 6, -8\rangle = \langle -82, -2, -8\rangle$ and the equation is $82x + 2y + 8z = 82 - 4 + 24 = 102$.

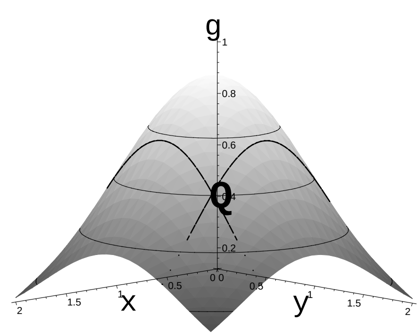
3. T2#4 F03 Directional Derivative. [Compare T2#4 S03 T2#5 F02] Find the directional derivative of $f(x, y, z) = x^3 + y^2 + z$ as you leave the point $P(3, 2, 1)$ heading in the direction of the point $Q(0, 6, 1)$.

   $\overrightarrow{PQ} = \langle -3, 4, 0\rangle$ which has length 5 so $\vec{u} = \langle -3/5, 4/5, 0\rangle$. $\nabla f = \left\langle 3x^2, 2y, 1\right\rangle$ and $\nabla f(P) = \langle 27, 4, 1\rangle$. Finally $f_{\vec{u}} = \nabla f \cdot \vec{u} = -81/5 + 16/5 + 0 = -65/5 = -13$.

4. T2#6 F02 Positive, Negative or Zero. [Compare T2#5 S03 T2#5 F03] The point P is on the contour graph of the function f (below left) and the point Q is on the surface of the graph of the function g (below right). Let $\vec{u}$ be the unit vector $\vec{u} = (-\vec{i} - \vec{j})/\sqrt{2}$.

   Find the sign (positive, negative or zero) of the partials of f: $f_x(P), f_y(P), f_{xx}(P), f_{yy}(P), f_{xy}(P)$ and the partials of g: $g_x(Q), g_y(Q), g_{xx}(Q)$ and the two directional derivatives $f_{\vec{u}}(P)$ and $g_{\vec{u}}(Q)$.





$f_x(P) < 0$, $f_{xx}(P) > 0$ as f decreasing and smiling as x increases. $f_y(P) > 0$, $f_{yy}(P) > 0$ as f decreasing and smiling as y increases. Eventually $f_{xy}(P) < 0$. Since $\vec{u}$ is parallel to the contours of f, $f_{\vec{u}}(P) = 0$. For g: $g_x(Q) < 0$, $g_{xx}(Q) < 0$, $g_y(Q) < 0$ and $g_{\vec{u}}(P) > 0$.

5. T2#3 F02 Taylor/Normal. For the function $f(x, y) = x^3 + xy + y^2$

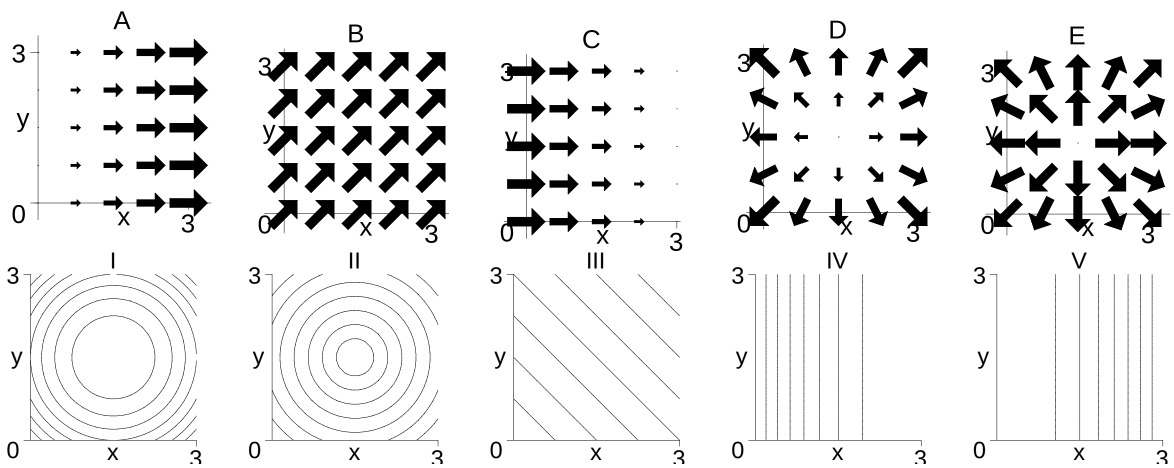
   (a) Compute the quadratic Taylor polynomial for f at the point $(-1, 2)$.

   (b) Compute the equation of the normal line to f at the point $(-1, 2)$.

   $\nabla f = \left\langle 3x^2 + y, x + 2y \right\rangle$, so $\nabla f(-1, 2) = \langle 5, 3\rangle$. We have $f(-1, 2) = 1$, $f_x(-1, 2) = 5$, $f_y(-1, 2) = 3$ (from gradient). For second derivatives $f_{xx} = 6x$, $f_{yy} = 2$ and $f_{xy} = 1$ so $f_{xx}(-1, 2) = -6$, $f_{yy}(-1, 2) = 2$ and $f_{xy}(-1, 2) = 1$.

   Taylor is $f(x, y) \approx 1+5(x+1)+3(y-2)+(-6(x+1)^2+2(x+1)(y-2)+2(y-2)^2)/2$ and the normal line is $x = -1 + 5t, y = 2 + 3t, z = 1 - t$.

6. T2#3 F03 Gradient plots matching. Match each of the gradient plots A–E to the matching contour plot among I–V.



   Obviously B and III match. The rest of the problem divides into 2 sets of two: A and C with IV and V; and D and E with I and II. For A and C the trick is to match the regions with greatest increase. So A is V and C is IV. For I and II we look for the uniform speed so E is II and D is I.

7. T2#6 S03 2D/3D tangents (This is problem #30 from H14.5). A differentiable function $f(x, y)$ has the property that $f(1, 2) = 7$ and $\nabla f(1, 2) = 2\vec{\mathbf{i}} - 5\vec{\mathbf{j}}$.

   (a) Find the equation of the tangent PLANE to the SURFACE $z = f(x, y)$ at the point $(1, 2, 7)$.

   (b) Find the equation of the tangent LINE to the level CURVE of f through the point $(1, 2)$.

   The normal to the tangent plane is $\langle 2, -5, -1\rangle$ so the equation is $2x - 5y - z = 2 - 10 - 7 = -15$. The normal to the tangent line is $\langle 2, -5\rangle$ so the equation is $2x - 5y = 2 - 10 = -8$.

8. T2#2 F00 Directional derivative application (also a problem from the text (second edition)). Suppose that $F(x, y, z) = x^2 + y^4 + x^2z^2$ gives the concentration of salt in a fluid at the point (x, y, z) and you are at the point $(-1, 1, 1)$.

   (a) In which direction (unit vector) should you move if you want the concentration to increase the fastest?

   (b) Suppose you start to move in the direction you found in part (a) at a speed of 4 units/sec. How fast is the concentration changing?

   $\nabla F = \left\langle 2x + 2xz^2, 4y^3, 2x^2z \right\rangle$ so $\nabla F(-1, 1, 1) = \langle -4, 4, 2\rangle$ which has length 6 so the direction of fastest increase is $\vec{u} = \langle -2/3, 2/3, 1/3\rangle$. The concentration would be increasing at $4F_{\vec{u}}$ or $4(8/3+8/3+2/3) = 24$ somethings per second.