Practice Mini-Test 6 – Calculus 3 – Spring 04

1. T3#1 F02 Find curl $\vec{F}$ and find div $\vec{G}$, if the vector field $\vec{F} = \langle xy^2z^3, 0, x^3y^2z \rangle$ and if the vector field $\vec{G} = \langle z\sin(x/y), z\sin(x/y), z\sin(x/y)\rangle$. [Compare T3#1 F03 T3#1 S02 T3#1 F02]

   curl $\vec{F} = \langle 2x^3yz, 3xy^2z^2 - 3x^2y^2z, -2xyz^3 \rangle$ and div $\vec{G} = \frac{z}{y}\cos\frac{x}{y} - \frac{xz}{y^2}\cos\frac{x}{y} + \sin\frac{x}{y}$

2. T3#2 F03 Vector, scalar or nonsense. $f = f(x, y, z)$ and $g = g(x, y, z)$ are scalar fields and $\vec{F} = \vec{F}(x, y, z)$ and $\vec{G} = \vec{G}(x, y, z)$ are vector fields. Determine if the given object is a scalar field, a vector field or nonsense. [Compare T3#2 S02,T3#3 F02]

A. $f + g$	B. $\vec{F} + \vec{G}$	C. $f + \vec{G}$	D. $\nabla \vec{F}$	E. div g
A. scalar	B. vector	C. nonsense	D. nonsense	E. nonsense
F. curl(curl $\vec{G}$)	G. curl(div $\vec{G}$)	H. div(curl $\vec{F}$)	I. grad(div $\vec{F}$)	J. div(grad g)
F. vector	G. nonsense	H. scalar	I. vector	J. scalar

3. T3#3 F03 Find a function f so that $\nabla f = \vec{F}$ and use it to compute the line intergal $\int_C \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F} = \langle y, x + 6y + z^2, 2yz + 1/z \rangle$ where C is some very complex counterclockwise corkscrew curve going from $(1, 3, 2)$ to $(5, -1, 3)$. [Compare T3#2 F02]

   $f(x, y, z) = xy + 3y^2 + yz^2 + \ln|z|$ works and the line integral is $f(5, -1, 3) - f(1, 3, 2) = (-5 + 3 - 9 + \ln(3)) - (3 + 27 + 12 + \ln(2)) = \ln\frac{3}{2} - 53$.

4. T3#5 F03 Compute the line intergal $\int_C \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F} = \langle y, z, x \rangle$ and C is the is the line from $(1, 3, 2)$ to $(5, -1, 3)$. [Compare T3#3 S02]

   $\vec{r}(t) = \langle 1 + 4t, 3 - 4t, 2 + t \rangle$ for $0 \le t \le 1$. $d\vec{r} = \langle 4, -4, 1 \rangle\, dt$ and the line integral is $\int_0^1 \langle 3 - 4t, 2 + t, 1 + 4t \rangle \cdot \langle 4, -4, 1 \rangle\, dt = \int_0^1 5 - 16t\, dt = -3$

5. T3#6 F03 Compute the flux of the vector field $\vec{F} = y\hat{\imath} + (1 + z)\hat{k}$ through the part of the plane $3x + 3y + z = 3$ oriented outward with $x \ge 0, y \ge 0, z \ge 0$ [Compare T3#10 F02]

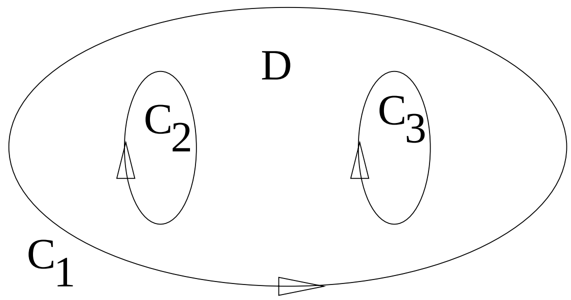
   Use the $z = f(x, y)$ form over the triangle formed by $(0, 0), (1, 0), (0, 1)$ in the xy-plane. $d\vec{A} = \langle 3, 3, 1 \rangle\, dA$ and $\vec{F}(\vec{r}) = \langle y, 0, 4 - 3x - 3y \rangle$.

$$\int_{x=0}^{1} \int_{y=0}^{1-x} 3y + 4 - 3x - 3y\, dy\, dx = \frac{3}{2}$$

6. T3#7 F03 Use your TI-89 to graph the closed curve C given by $\vec{r}(t) = \langle \cos^3 t, \sin^3 t \rangle$ for $0 \le t \le 2\pi$. For the vector field $\vec{F} = \langle 0, x \rangle$ write and simplify the line integeral $\oint_C \vec{F} \cdot d\vec{r}$ down to a regular integral in t. Use your TI-89 to evaluate this integral. Finally use Green's theorem to explain why your answer is the area of the region enclosed by C. [Compare T3#10 S03 T3#9 F02 T3#9 S02]

   The plot is a diamond with sides "that curve inward" (toward the origin). We have $d\vec{r} = \langle -3\cos^2 t \sin t, 3\sin^2 t \cos t \rangle\, dt$, and $\vec{F}(\vec{r}) = \langle 0, \cos^3 t \rangle$ so line integral is $\int_0^{2\pi} 3\cos^4 t \sin^2 t\, dt = 3\pi/8$ by the calculator. Green's theorem says this value is $Q_x - P_y = 1$ (double) integrated over the region, which is the area of the region.

7. T3#8 F03 The domain D of area 13 is the region inside C_1 but outside C_2 and C_3 (see below left) and $\vec{F} = \left\langle 7x^2 - 5y, 3x + 11y^2 \right\rangle$, find the line integral $\oint_{C_1} \vec{F} \cdot d\vec{r}$ if $\oint_{C_2} \vec{F} \cdot d\vec{r} = -15$ and $\oint_{C_3} \vec{F} \cdot d\vec{r} = -25$.



Greens theorem again, $Q_x - P_y = 3 - (-5) = 8$ so the double integral over D is $8 \cdot 13$ and this is the sum of the three line integrals over C_1 plus C_2 plus C_3 so the integral over $C_1 = 144$.

8. T3#5 S03 The vector field $\vec{F} = \langle 5, 2, 3\rangle$ compute the flux of $\vec{F}$ over each of the surfaces S described below without evaluation any integrals. When there is a choice of normal, pick the normal whose dot product with $\vec{F}$ is positive.

   (a) S is the rectangle $\{(0, y, z), 1 \le y \le 3, 0.5 \le z \le 1\}$.

   (b) S is a circle in the xz-plane with radius 5.

   (c) S is a pentagon of area P with normal in the direction of $\langle 5, -5, -5\rangle$

   [Compare T3#5 F02 T3#5 S02]

   (a) The unit normal is $\vec{i}$, the area 1 and the flux is 5. (b) The unit normal is $\vec{j}$, the area 25π and the flux is 50π. (c) The normal given has length $5\sqrt{3}$, the area is P, and the flux is $P(25 - 10 - 15)/(5\sqrt{3}) = 0$.

9. T3#6 S03 Use a theorem to show that if V is a 3D region in space with outward oriented boundry given by S then

$$\iint_S \langle x, y, z\rangle \cdot d\vec{A} = 3 \text{ Volume}\,(V)$$

   [Compare T3#7 F02]

   Since $\text{div}\langle x, y, z\rangle = 3$, Gauss's Thereom says the flux $= \iiint_V 3\, dV = 3\text{Volume}(V)$

10. T3#7 S03 Read this problem carefully. You are to sketch both of the vector fields and both the curves and explain the answers below but **NOT** by computing the line integrals. The two vector fields, let $\vec{F} = \langle x, 0\rangle$ and let $\vec{G} = \langle 0, x\rangle$. The two curves, let C_1 be the straight line from $(1, 1)$ to $(3, 1)$ and let C_2 be the circle or radius $2/\sqrt{\pi}$ centered at $(1, 1)$ oriented counterclockwise. The following integrals were all be computed without integrating a single line integral, either by the fundemental theorem of calculus for line integrals, or by geometry, or by Green's theorem. Explain how each integral below was obtained without doing a single line integral. [Hint: start by checking if $\vec{F}$ or $\vec{G}$ is a gradient vector field.]

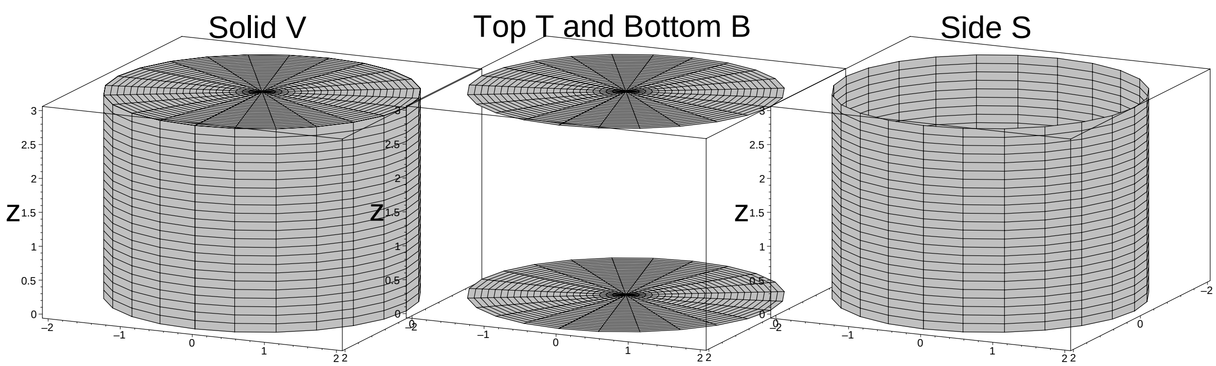
$$(A) \int_{C_1} \vec{F} \cdot\, d\vec{r} = 4 \qquad (B) \int_{C_1} \vec{G} \cdot\, d\vec{r} = 0 \qquad (C) \oint_{C_2} \vec{F} \cdot\, d\vec{r} = 0 \qquad (D) \oint_{C_2} \vec{G} \cdot\, d\vec{r} = 4$$

   Note $\vec{F}$ is conservative, it is ∇f where $f(x, y) = x^2/2$ This means the integral around C_2 is 0 and the integral along C_1 is $f(3, 1) - f(1, 1) = 9/2 - 1/2 = 4$. On the other hand, $\vec{G}$ is perpendicular to C_1 so the integral along C_1 is zero. The integral around the close curve C_2 can be found by Green's Theorem to be area of the disk since $Q_x - P_y = 1 - 0 = 1$. This area is $\pi(2/\sqrt{\pi})^2 = 4$.

11. T3#10 F03 Divergence Theorem. Let V be volume inside the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3$. The boundry of V is oriented with outward pointing normal and divided into pieces as follows. Let B be the surface on the bottom of the cylinder, T be the surface at the top of the cylinder and S be the surface at the side of the cylinder. (See graphs below.) The vector field is $\vec{F} = \langle xy^2, xz^2, x^2z \rangle$ compute the following three integrals, the first without integrating, the last two by converting to polar or cylindrical coordinates.

$$\iint_B \vec{F} \cdot d\vec{A} \qquad \iint_T \vec{F} \cdot d\vec{A} \qquad \iiint_V \text{div } \vec{F} \, dV$$

Using the Divergence Theorem you can use the information above to find the value of the flux over S. Do so, and also write the and simplify the flux integral over S. Do **NOT** evaluate this last integral. [Well, using the TI-89 to evaluate the integral would be a way of checking your answer.]



We did this one in class. Flux over B was 0, over T was 12π, the triple integral had value 24π so the flux over S was 12π.