Theorem. Every matrix $A$, *decomposes nicely* into generalized eigenspaces.

1. Find a polynomial $p(\lambda)$ so that $p(A)$ has a non-trivial kernel

   (a) The characteristic polynomial works, but it requires the determinant. the method below is actually "easier".

   (b) Instead let $\vec{v}$ be any non-zero vector. Since $\{A^0\vec{v}, A^1\vec{v}, \ldots A^n\vec{v}\}$ must be linearly dependent, there are scalars $c_i$ so that

   $$c_0A^0\vec{v} + c_1A^1\vec{v} + \cdots c_nA^n\vec{v} = 0$$

   Let $p(\lambda) = c_0 + c_1\lambda + \cdots c_n\lambda^n$, and note $p(A)$ maps $\vec{v}$ to the zero vector.

2. The existence of an eigenvalue $\lambda$.

   (a) The fundament theorem of algebra says $p(x) = a(x - \lambda_1)\cdots(x - \lambda_n)$ for some $a$ and roots $\lambda_i$.

   (b) $p(A)\vec{v} = a(A - \lambda_1 I)\cdots(A - \lambda_n I)\vec{v}$ and one of the $(A - \lambda_i)$ does not have a trivial kernel so that $\lambda_i$ is an eigenvalve.

3. The generalized eigenspace for $\lambda$ is $N = \ker(A - \lambda I)^n$ and its "complement" is $C = \text{range}(A - \lambda I)^n$

   (a) Eventually $N = \ker(A-\lambda I)^k = \ker(A-\lambda I)^{k+1}$ and this will be the generalized eigenspace. Since $\ker(A - \lambda I)^k$ increases by at least one dimension each time $k$ increases by one, $k \leq n$.

4. Both subspaces $N$ and $C$ are invariant under $A$.

   (a) if $AB = BA$ then $B(\ker A) \subset \ker A$ and $B(\text{range}\, A) \subset \text{range}\, A$.

      i. If $\vec{x} \in \ker A$, then $AB\vec{x} = B(A\vec{x} = B0 = 0$ so $B\vec{x} \in \ker A$.

      ii. If $\vec{x} \in \text{range}\, A$, then there is $\vec{y}$ so $\vec{x} = A\vec{y}$. Thus $B\vec{x} = BA\vec{y} = A(B\vec{y})$ must be in range $A$.

   (b) $A(A - \lambda I)^n = (A - \lambda I)^n A$

5. $A|_C$ does not have $\lambda$ as an eigenvalue.

   (a) Any such eigenvector would already be in $N$.

6. Induction, Continue applying the result to $A|_C$ which must have smaller dimension.