More problems on Linear Systems

1. Solve the dynamic system $x_{k+1} = Ax_k$, $x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ if $A = \begin{bmatrix} -1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$. What is dynamic behavior?

2. Solve the DE system $x'(t) = Ax(t)$, $x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ if $A = \begin{bmatrix} -1/2 & 1 \\ 0 & 1/2 \end{bmatrix}$. What is dynamic behavior?

3. Solve the dynamic system $x_{k+1} = Ax_k$, $x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ if $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$. Write your answer without complex numbers. What is dynamic behavior?

4. Solve the DE system $x'(t) = Ax(t)$, $x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ if $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$. Write your answer without complex numbers. What is dynamic behavior?

5. Solve the dynamic system $x_{k+1} = Ax_k$, $x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ if $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. What is dynamic behavior?

6. Solve the DE system $x'(t) = Ax(t)$, $x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ if $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. What is dynamic behavior?

Solutions

1. Eventually A has eigenvalues/eigenvectors $\lambda_1 = 1/2$, $\vec{v}_1 = [1,1]^T$ and $\lambda_2 = -1/2$, $\vec{v}_2 = [1,0]^T$. Since $x_0 = \vec{v}_1 - 2\vec{v}_2$, the solution is $x_k = (1/2)^k\vec{v}_1 - 2(-1/2)^k\vec{v}_2$. This shrinks to the origin.

2. We have the same eigenvalues/eigenvectors and now $x(0) = \vec{v}_1 - 2\vec{v}_2$, the solution is $x(t) = e^{t/2}\vec{v}_1 - 2e^{-t/2}\vec{v}_2$. This is a saddle point.

3. Eventually A has eigenvalues/eigenvectors $\lambda_1 = 4+i$, $\vec{v}_1 = [1+i,1]^T$ and $\lambda_2 = 4-i$, $\vec{v}_2 = [1-i,1]^T$. Since $x_0 = (1/2+i)\vec{v}_1 + (1/2-i)\vec{v}_2$ the solution is $x_k = (4+i)^k(1/2+i)\vec{v}_1 + (4-i)^k(1/2-i)\vec{v}_2$. This spirals out.

   Now to remove the complex numbers. No please don't use the binomial theorem. Use change of variables $y_k = P^{-1}x_k$ or $x_k = Py_k$ where $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 4 & -1 \\ 1 & 4 \end{bmatrix} = \sqrt{17}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $C^k = 17^{k/2}\begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$, and $P^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

   Now $y_0 = P^{-1}x_0 = [1,2]^T$, $y_k = C^k y_0 = 17^{k/2}(\begin{bmatrix} \cos k\theta - 2\sin k\theta \\ \sin k\theta + 2\cos k\theta \end{bmatrix}$ and so $x_k = Py_k = 17^{k/2}\begin{bmatrix} -\cos k\theta - 3\sin k\theta \\ \cos k\theta - 2\sin k\theta \end{bmatrix}$

   Alternately, we observe both u and v are independent solutions to $y_{k+1} = Cy_k$ with $u_0 = [1,0]^T$ and $v_0 = [0,1]^T$ where. $u_k = 17^{k/2}\begin{bmatrix} \cos k\theta \\ \sin k\theta \end{bmatrix}$ $v_k = 17^{k/2}\begin{bmatrix} -\sin k\theta \\ \cos k\theta \end{bmatrix}$. Since we want $y_0 = [1,2]^T$, pick $y_k = 1u_k + 2v_k$ and $x_k = P(u_k + 2v_k) = Pu_k + 2Pv_k$

4. We have the same eigenvalues/eigenvectors and now $x(0) = (1/2+i)\vec{v}_1 + (1/2-i)\vec{v}_2$ the solution is $x(t) = (1/2+i)e^{t(4+i)}\vec{v}_1 + (1/2-i)e^{t(4-i)}\vec{v}_2$ This also spirals outward.

   To remove the complex numbers we convert to the real and imaginary parts of $\vec{w} = e^{4t}(\cos t + i\sin t)\vec{v}_1$ $u(t) = \operatorname{Re}\vec{w} = \begin{bmatrix} e^{4t}\cos t - e^{4t}\sin t \\ e^{4t}\cos t \end{bmatrix}$ $v(t) = \operatorname{Im}\vec{w} = \begin{bmatrix} e^{4t}\cos t + e^{4t}\sin t \\ e^{4t}\sin t \end{bmatrix}$ Are both solutions to $x'(t) = Ax(t)$ with $u(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Eventually $x(t) = u(t) - 2v(t) = \begin{bmatrix} -e^{4t}\cos t - 3e^{4t}\sin t \\ e^{4t}\cos t - 2e^{4t}\sin t \end{bmatrix}$

5. Eventually A has only the repeated eigenvalue 1, eigenvector $\vec{v}_1 = [1,1]^T$ and generalized eigenvector $\vec{v}_2 = [1,0]^T$ $x_0 = \vec{v}_1 - 2\vec{v}_2$, the solution is $x_k = (1)^k\vec{v}_1 - 2k(1)^{k-1}\vec{v}_1 - 2(1)^k\vec{v}_2 = \vec{v}_1 - 2k\vec{v}_1 - 2\vec{v}_2$. This expands in the $-\vec{v}_1$ direction.

6. We have the same eigenvalues/eigenvectors and now $x(0) = \vec{v}_1 - 2\vec{v}_2$, the solution is $x(t) = e^t\vec{v}_1 - 2te^t\vec{v}_1 - 2e^t\vec{v}_2$. This expands.