Linear Programming §9.1 #15 geometric and via simplex method.

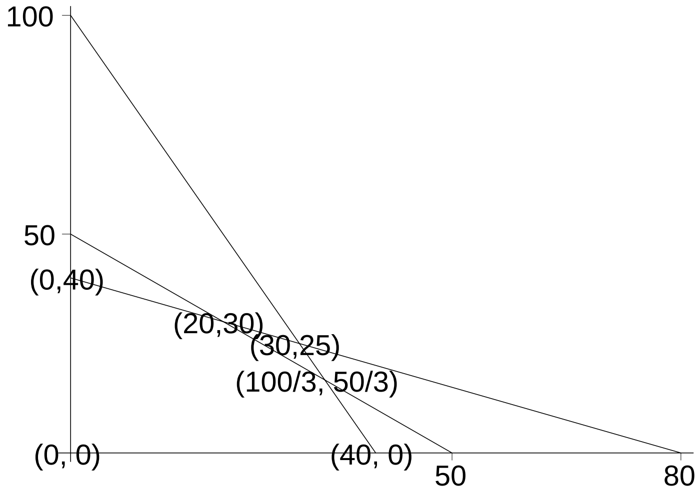
1. Problem statement: Let x_1 be the number of Widgets and x_2 the number of Whammies, eventually we reduced the problem to

$$\max\{20x_1 + 26x_2\}$$

given

$$5x_1 + 2x_2 \leq 200$$
$$x_1 + 2x_2 \leq 80$$
$$x_1 + x_2 \leq 50$$
$$x_1 \geq 0; \quad x_2 \geq 0$$

2. Geometric Solution, plot the feasible region $\mathcal{F}$



The extreme points were found by setting the inequalities to equalities and solving them a pair at a time. Note $(30, 25)$ is outside the feasible region. By checking the *extreme points* we see the maximum value of \$1180 occurs when $x_1 = 20$ and $x_2 = 30$.

x	$f(x)$
$(0,0)$	0
$(0,40)$	1040
$(40,0)$	800
$(20,30)$	1180
$(100/3, 50/3)$	1100

3. Simplex Method

   (a) Add slack variables, we increase the number of x_i variables adding one for each inequality changing it into an equality. This changes the problem to

$$\max\{20x_1 + 26x_2\}$$

given

$$5x_1 + 2x_2 + x_3 = 200$$
$$x_1 + 2x_2 + x_4 = 80$$
$$x_1 + x_2 + x_5 = 50$$
$$x_1 \geq 0; \quad x_2 \geq 0; \quad x_3 \geq 0; \quad x_4 \geq 0; \quad x_5 \geq 0$$

(b) Add a variable $M = 20x_1 + 26x_2$ for the function we are tying to optimize. Write this as $-20x_1 - 26x_2 + M = 0$. Treat M like it was x_6 in some sense.

(c) The initial simplex tableau. The variables $x_1 = x_2 = 0$ are "out", the variable $x_3 = 200$, $x_4 = 80$ and $x_5 = 50$ are "in" and $M = 0$. Bringing either x_1 or x_2 in will increase M so we are not optimal, geometrically we are at the extreme point $(0, 0)$.

x_1	x_2	x_3	x_4	x_5	M	b
5	2	1	0	0	0	200
1	2	0	1	0	0	80
1	1	0	0	1	0	50
−20	−26	0	0	0	1	0

(d) We need to decide what variable to bring in and which row to pivot on. We pick x_2 because $26 > 20$ so changes in x_2 have more effect than changes in x_1. To keep us inside the feasible region, we pick the row with both positive $a_{i,2}$ and smallest $b_i/a_{i,2}$. Here the numbers are $200/2 = 100$, $80/2 = 40$ and $50/1 = 50$ so we pivot about row 2. (Never pivot on the bottom row.)

x_1	x_2	x_3	x_4	x_5	M	b
4	0	1	−1	0	0	120
1/2	1	0	1/2	0	0	40
1/2	0	0	−1/2	1	0	10
−7	0	0	13	0	1	1040

Now x_1 and x_4 are out and $x_2 = 40$, $x_3 = 120$, $x_5 = 10$ $M = 1040$. Geometrically we are at the extreme point $(0, 40)$. We are not done, as bringing x_1 in will increase M.

(e) There is only variable to bring in which will increase M. Which row? $120/4 = 30$, $40/(1/2) = 80$, and $10/(1/2) = 20$ so we pivot on row 3.

x_1	x_2	x_3	x_4	x_5	M	b
0	0	1	3	−8	0	40
0	1	0	1	−1	0	30
1	0	0	−1	2	0	20
0	0	0	6	14	1	1180

Now x_4 and x_5 are out and $x_1 = 20$, $x_2 = 30$, $x_3 = 40$ $M = 1180$. Geometrically we are at the extreme point $(20, 30)$. There is no variable which will increase M so we are at the optimal solution.

4. Scilab

```
-->p = [20;26]; // profit
-->b = [200;80;50]; // b
-->a = [5 2; 1 2; 1 1]; // ax <= b
-->[x, lagr, f] = linpro(-p, a,b,[0,0]',[]) // min's instead of max
 f  = - 1180.   // max with minus sign
 lagr  =
!   0.  !
!   0.  !
!   0.  !
!   6.  !
!   14. !
 x  =         // extreme point
!   20. !
!   30. !
```