Linear Programming §9.1 #15 geometric and via simplex method.

1. Problem statement: Let $x_1$ be the number of Widgets and $x_2$ the number of Whammies, eventually we reduced the problem to

$$\max \{20x_1 + 26x_2\}$$

given

$$5x_1 + 2x_2 \leq 200$$
$$x_1 + 2x_2 \leq 80$$
$$x_1 + x_2 \leq 50$$
$$x_1 \geq 0; \quad x_2 \geq 0$$

2. Geometric Solution, plot the feasible region $F$

The extreme points were found by setting the inequalities to equalities and solving them a pair at a time. Note $(30, 25)$ is outside the feasible region. By checking the extreme points we see the maximum value of $\$1180$ occurs when $x_1 = 20$ and $x_2 = 30$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0, 0)$</td>
<td>0</td>
</tr>
<tr>
<td>$(0, 40)$</td>
<td>1040</td>
</tr>
<tr>
<td>$(40, 0)$</td>
<td>800</td>
</tr>
<tr>
<td>$(20, 30)$</td>
<td>1180</td>
</tr>
<tr>
<td>$(100/3, 50/3)$</td>
<td>1100</td>
</tr>
</tbody>
</table>

3. Simplex Method

(a) Add slack variables, we increase the number of $x_i$ variables adding one for each inequality changing it into an equality. This changes the problem to

$$\max \{20x_1 + 26x_2\}$$

given

$$5x_1 + 2x_2 + x_3 = 200$$
$$x_1 + 2x_2 + x_4 = 80$$
$$x_1 + x_2 + x_5 = 50$$
$$x_1 \geq 0; \quad x_2 \geq 0; \quad x_3 \geq 0; \quad x_4 \geq 0; \quad x_5 \geq 0$$
(b) Add a variable \(M = 20x_1 + 26x_2\) for the function we are tying to optimize. Write this as 
\[-20x_1 - 26x_2 + M = 0\]. Treat \(M\) like it was \(x_6\) in some sense.

(c) The initial simplex tableau. The variables \(x_1 = x_2 = 0\) are “out”, the variable \(x_3 = 200, x_4 = 80\) and \(x_5 = 50\) are “in” and \(M = 0\). Bringing either \(x_1\) or \(x_2\) in will increase \(M\) so we are not optimal, geometrically we are at the extreme point \((0, 0)\).

\[
\begin{bmatrix}
  x_1 & x_2 & x_3 & x_4 & x_5 & M & b \\
  5 & 2 & 1 & 0 & 0 & 0 & 200 \\
  1 & 2 & 0 & 1 & 0 & 0 & 80 \\
  1 & 1 & 0 & 0 & 1 & 0 & 50 \\
  -20 & -26 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

(d) We need to decide what variable to bring in and which row to pivot on. We pick \(x_2\) because \(26 > 20\) so changes in \(x_2\) have more effect than changes in \(x_1\). To keep us inside the feasible region, we pick the row with both positive \(a_{i,2}\) and smallest \(b_i/a_{i,2}\). Here the numbers are \(200/2 = 100, 80/2 = 40\) and \(50/1 = 50\) so we pivot about row 2. (Never pivot on the bottom row.)

\[
\begin{bmatrix}
  x_1 & x_2 & x_3 & x_4 & x_5 & M & b \\
  4 & 0 & 1 & -1 & 0 & 0 & 120 \\
  1/2 & 1 & 0 & 1/2 & 0 & 0 & 40 \\
  1/2 & 0 & 0 & -1/2 & 1 & 0 & 10 \\
  -7 & 0 & 0 & 13 & 0 & 1 & 1040 \\
\end{bmatrix}
\]

Now \(x_1\) and \(x_4\) are out and \(x_2 = 40, x_3 = 120, x_5 = 10 M = 1040\). Geometrically we are at the extreme point \((0, 40)\). We are not done, as bringing \(x_1\) in will increase \(M\).

(e) There is only variable to bring in which will increase \(M\). Which row? \(120/4 = 30, 40/(1/2) = 80,\) and \(10/(1/2) = 20\) so we pivot on row 3.

\[
\begin{bmatrix}
  x_1 & x_2 & x_3 & x_4 & x_5 & M & b \\
  0 & 0 & 1 & 3 & -8 & 0 & 40 \\
  0 & 1 & 0 & 1 & -1 & 0 & 30 \\
  1 & 0 & 0 & -1 & 2 & 0 & 20 \\
  0 & 0 & 0 & 6 & 14 & 1 & 1180 \\
\end{bmatrix}
\]

Now \(x_4\) and \(x_5\) are out and \(x_1 = 20, x_2 = 30, x_3 = 40 M = 1180\). Geometrically we are at the extreme point \((20, 30)\). There is no variable which will increase \(M\) so we are at the optimal solution.

4. Scilab

```scilab
-->p = [20;26]; // profit
-->b = [200;80;50]; // b
-->a = [5 2; 1 2; 1 1]; // ax <= b
-->[x, lagr, f] = linpro(-p, a,b,[0,0]',[]); // min’s instead of max
f = -1180. // max with minus sign
lagr =
  ! 0. !
  ! 0. !
  ! 0. !
  ! 6. !
  ! 14. !
x = // extreme point
  ! 20. !
  ! 30. !
```