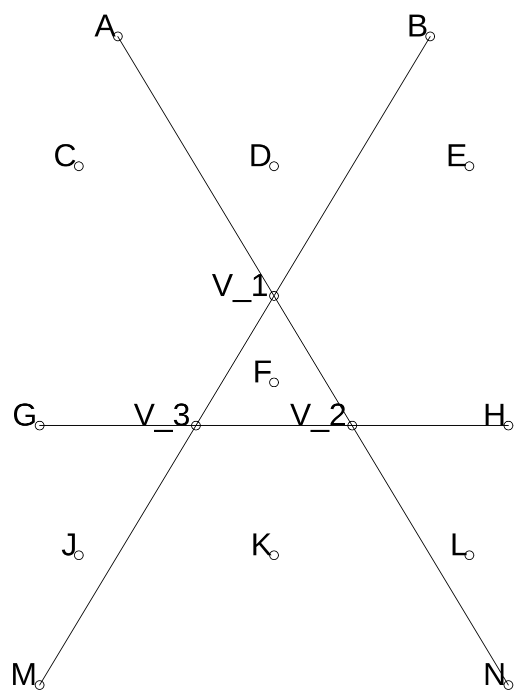
Show **ALL** work for credit; Give **EXACT** answers when possible; Start each problem on a **SEPARATE** page; Use only **ONE** side of each page; Be neat; Leave margins on the left and top for the **STAPLE**; Calculators can be used for graphing and calculating only; Nothing written on this page will be graded;

1. Let $Q(x)$ be the quadratic form $x_1^2 + 4x_1x_2 + x_2^2$ on $\mathbb{R}^2$
   (a) Find a symmetric matrix A so that $Q(x) = x^T Ax$.
   (b) Is A positive definite, positive semi-definite, negative definite, negative semi-definite or indefinite?
   (c) What is the maximum value of $Q(x)$ on the sphere $x^T x = 1$?
   (d) What is the minimum value of $Q(x)$ on the sphere $x^T x = 1$?
   (e) Find an orthogonal matrix P such that the change of variable $x = Py$ transforms $x^T Ax$ into a quadratic form with no cross product terms.



2. Barycentric coordinates. The picture above shows three vectors v_1, v_2 and v_3 in the plane $\mathbb{R}^2$ and a number of other points ($A - N$ with I missing) and lines. Since $\{v_1, v_2, v_3\}$ is affinely independent, each point P on the plane as unique barycentric coordinates (c_1, c_2, c_3) so that P is the affine combination $c_1v_1 + c_2v_2 + c_3v_3$.
   (a) Which point from A through N has coordinates $(0, +, -)$ (that is $c_1 = 0$, $c_2 > 0$ and $c_3 < 0$)?
   (b) Which point from A through N has coordinates $(+, +, +)$?
   (c) What two element subset S of $\{v_1, v_2, v_3\}$ has the line $AN \subset \text{aff } S$?
   (d) What signs are the affine coordinates of the point D?
   (e) Which of the points from A through N have a $c_2 > 0$.

There is more test on the other side

3. True or False.

(a) Every convex set is affine.

(b) If $\{v_1, v_2, \ldots v_n\}$ is affinely dependent then $\{v_2 - v_1, v_3 - v_1, \ldots v_n - v_1\}$ is linearly dependent.

(c) If $A = U\Sigma V^T$ is the singular value decomposition of a rank k operator, then first k columns of U is an orthonormal basis for $\operatorname{Col} A$.

(d) The set $\{(x, y, z) \in \mathbb{R}^3 : z \geq 5\}$ is affine.

(e) $2 + 5i$ can be an eigenvalue to for a symmetric $n \times n$ matrix.

(f) There is a symmetric matrix A with eigenvector $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ for the eigenvalue 5 and the eigenvector $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ for the eigenvalue 3.

(g) In a Principal Component Analysis of $X_1 \ldots X_N$ in $\mathbb{R}^3$, the 3×3 covariance matrix has eigenvalues 9, 2.5 and 0.5 with eigenvectors $\begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$, then 75% of the variance is in the "index" of performance given by $(x_1 - x_2 + x_3)/\sqrt{3}$.

(h) The $\operatorname{conv}\{(x, y) : y = e^{-x}, x \geq 0\} = \{(0, 1)\} \cup \{(x, y) : e^{-x} \leq y \leq 1, x \geq 0\}$.

(i) The hyperplane $x + y + z = 5$ in $\mathbb{R}^3$ strictly separates the origin $A = \{(0, 0, 0)\}$ from the sphere $B = \{(x, y, z) : x^2 + y^2 + z^2 = 500\}$

(j) If the linear functional $f(x) = [1, 1, 1]\, x$, and vectors $v_1 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, then $\operatorname{conv}\{v_1, v_2, v_3\} \subset [f : 5]$.

4. Find the singular value decomposition for the matrix A, that is find the orthogonal matrices U and V and a diagonal-like matrix Σ so that $A = U\Sigma V^T$, when

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$