GOING AROUND IN CIRCLES
SAMPLE LAB WRITE UP

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ABSTRACT. This is a sample lab write up

Phase plot Euler method stepsize = 1/10

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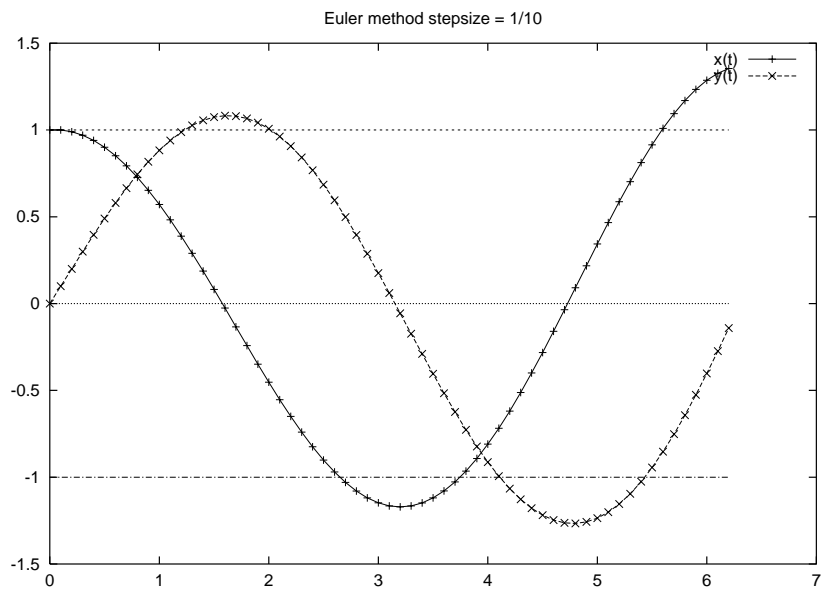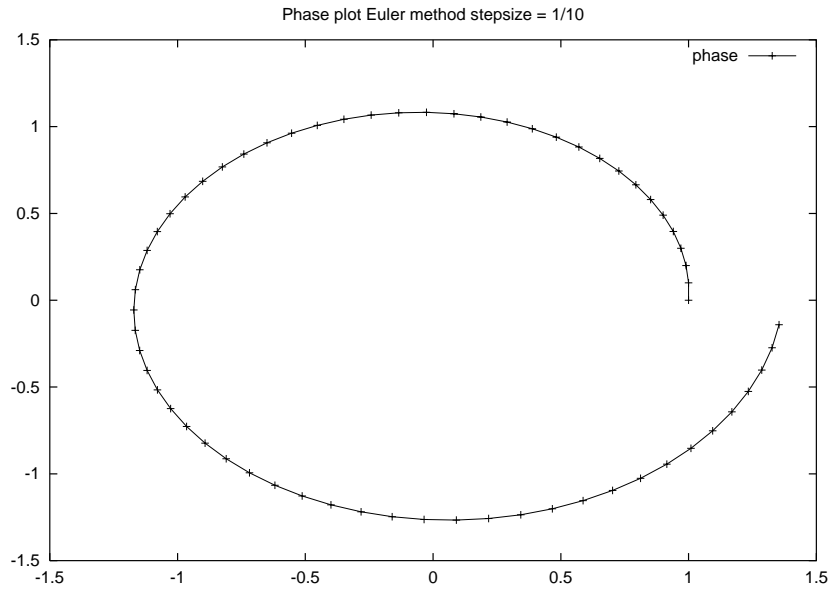
INTRODUCTION

Our problem is to compare two common numerical methods which are available on most technologies on a sample ODE. The two methods are often called Euler's method and Runga-Kutta, but they can have other names too.

Our sample system is the ODE

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x$$

and we will be interested in the initial value problem with $x(0) = 1$ and $y(0) = 0$.

Numerical methods for ODE's have a stepsize parameter which we will vary. We will use a base case stepsize of 0.1, and compare this to a stepsize of 0.05 and 0.2. We will base our error messurement at $t = 6.2 \approx 2\pi$, by how far the numerical answer is from the exact value.

The initial problem has an analytic solution given by

$$x(t) = \cos(t)$$

$$y(t) = \sin(t)$$

and we want to see plots of both x and y versus time t to see how these change with time. Also we want a phase diagram, the plot of the parameteric equations $(x(t), y(t))$. We will also want to know how the error varies with the stepsize and the method from the exact.

$$x(6.2) = \cos(6.2) = 0.9965420970$$

$$y(6.2) = \sin(6.2) = -0.08308940282$$

SOLUTION WITH THE TI-89

We assume that we are differential equation mode so all we need to do is set up the equations, Δt and solution method. We need a way to transfer the data from the calculator to the computer. One can always type the data in, but newer calculators plug directly into the computer. For example here are some screen dumps.
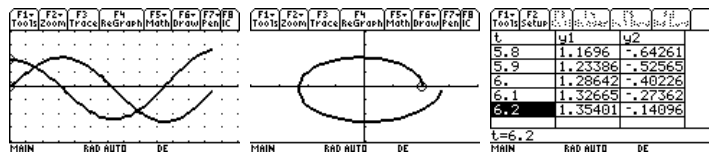
To start, we made sure we are in diff equations mode for graph, radian mode for angle, auto mode for exact/approx. We need to enter our system of equations into the green diamond $Y =$ screen. We let $y1$ be x and $y2$ by y, our IVP becomes

$$t0 = 0.0$$

$$\checkmark y1' = -y2$$

$$yi1 = 1.0$$

$$\checkmark y2' = y1$$

$$yi2 = 0.0$$

The next step is tricky, while in the $Y =$ screen do green diamond — where — is the symbol next to 7. This pulls up the graph mode screen which is important in DE mode. First we

will do solution method EULER. (The RK mode is much prefered for reasons which will become clear.) Second pick FLDOFF for fields (stands for field off). Picking the wrong field can prevent the graph from being drawn. A system of two equations can have a direction field or field off. A single equation can have a slope field or field off. A system of more than two equations must have the field off.

Next we set the window screen (green diamond WINDOW) to set the ranges:

$$t0 = 0$$

$$tmax = 6.2$$

$$tstep = 0.1$$

$$xmin = 0$$

$$xmax = 7$$

$$ymin = -2$$

$$ymax = 2$$

We are ready to graph, green diamond GRAPH and we see a screen much like the screen dump above. The screen dump was enhanced by going back to the green diamond — graph mode screen and turning on the grid which added the dots.

Next we want to see the actual data. We will use the table. Do green diamond TblSet. Make tblStart 0 (t0) and Δtbl 0.1 (tstep) and Independent point to auto. Now green diamond TABLE will show you the values that got ploted in the graph. I saved this into a memory variable using the BldData comand and suck the resulting data to the computer.

Next we want the phase diagram in the middle. Go back to $Y =$ and hit F7 AXES to change axes from time to custom. Select $y1$ for x and $y2$ for y. Change the window so x and y range from -2 to 2. Now green diamond GRAPH should show a spiral.

I used the BldData command to create xyzt which contained the graphing data (with a small error) that I uploaded to computer to use with gnuplot to produce a better looking pictures below and get the data directly into the calculator.
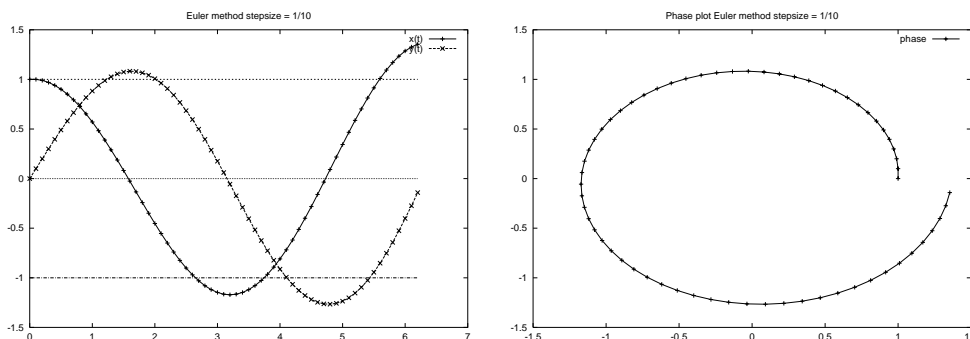
t	y1	y2
0	1	0
0.1	1	0.1
0.2	0.99	0.2
0.3	0.97	0.299
0.4	0.9401	0.396
0.5	0.9005	0.49001
0.6	0.851499	0.58006
0.7	0.793493	0.6652099
0.8	0.72697201	0.7445592
0.9	0.65251609	0.817256401
1	0.5707904498999	0.88250801
1.1	0.4825396489	0.9395870549899999
1.2	0.3885809434009	0.9878410198799999
1.3	0.2897968414129	1.0266991142201
1.4	0.1871269299909	1.0556787983614
1.5	0.0815590501548	1.0743914913605
⋮	⋮	⋮
5	0.3433546555151	-1.2356129662561
5.1	0.4669159521407	-1.2012775007046
5.2	0.5870437022111999	-1.1545859054905
5.3	0.7025022927602	-1.0958815352694
5.4	0.8120904462872	-1.0256313059934
5.5	0.9146535768865	-0.9444222613645999
5.6	1.009095803023	-0.852956903676
5.7	1.0943914933906	-0.7520473233736999
5.8	1.169596225728	-0.6426081740346
5.9	1.2338570431315	-0.5256485514618
6	1.2864218982777	-0.4022628471487
6.1	1.3266481829926	-0.2736206573209
6.2	1.3540102487247	-0.1409558390216

Tables of Error vs Method and Stepsize

First Euler's method, which has $O(h)$ error estimate and indeed halving the stepsize does halve the error.

Δt	$x(6.2)$	$y(6.2)$	Error
exact	0.9965420970	-0.08308940282	0
0.2	1.8119889011614	-0.2997159196565	0.8437301334
0.1	1.3540102487247	-0.1495583902168	0.3635953878
0.05	1.1628793365276	-0.10300179595918	0.1675248662

Second is the RK method (Runge-Kutta), which has $O(h^4)$ error estimate. But look we get the same values for all stepsizes. The problem is so simple that RK is very good at obtaining a reasonable approximation.

Δt	$x(6.2)$	$y(6.2)$	Error
exact	0.9965420970	-0.08308940282	0
0.2	0.99038419140595	-0.081040033433743	0.006489970438
0.1	0.99038419140595	-0.081040033433743	0.006489970438
0.05	0.99038419140595	-0.081040033433743	0.006489970438

Your Lab 1 Assignment

This is the SIR model of a biological epidemic:

- In a total population of n individuals there are, at any time t,
- $y(t)$ infectious carriers of a contagious disease,
- $x(t)$ members of the population who are susceptible to the disease, and
- $z(t)$ individuals who are recovered and immune.
- It is clear $x(t) + y(t) + z(t) = n$ for all $t \geq 0$.

The following systems of equations model the epidemic

$$\frac{dx}{dt} = -\lambda x(t) y(t)$$

$$\frac{dy}{dt} = \lambda x(t) y(t) - \mu y(t)$$

$$\frac{dz}{dt} = \mu y(t)$$

(1) Using the parameters t is time in days, $\lambda = 0.001$, $\mu = 1/14$, $n = 1000$ and the initial conditions $x(0) = 900$, $y(0) = 10$, $z(0) = 90$ and using some form of technology plot $x(t)$, $y(t)$, $z(t)$ and dy/dt on a graph for 15 days using a step size of 0.1 days.

(2) Explain what technology you used and the commands used to generate the data.

(3) Study the effect of the parameter λ by comparing the plot of $y(t)$ above with the plots of other values of λ, and filling out a table like the one below

λ	peak $y(t)$	peak time	limiting $x(t)$
0.0001	?	?	?
0.001	?	?	?
0.002	?	?	?

(4) Study the effect of the parameter μ by comparing the plot of $y(t)$ above with the plots of other values of μ, and by filling out a table like the one below.

μ	peak $y(t)$	peak time	limiting $x(t)$
1/21	?	?	?
1/14	?	?	?
1/7	?	?	?