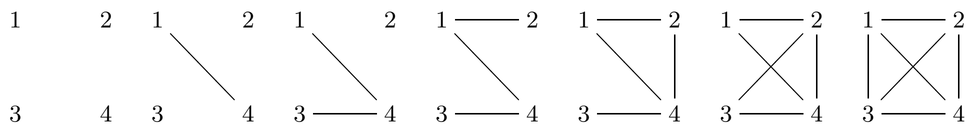
EVOLUTION OF RANDOM GRAPHS

MAD 5932 SUMMER 2006

Abstract. Our project will be to use scilab (a matlab clone) to explore random graphs. In particular, we study the evolution of the graphs on n vertices as we randomly add edges. We go from 0 edges to K_n, looking for the *onsets of phase changes*.



What is a random graph? What is an evolution?

Given a number of vertices n and an edge probability p we can generate a *random graph* G on the n vertices $\{1, 2, ..n\}$ by consider each pair of vertices, u and v, and randomly picking a real number between 0 to 1. If the random number is greater than $1 - p$ we add the edge uv to G. The expected number of edges is $p\binom{n}{2}$, but of course this is usually not the exact number of edges in G.

Given the set of n vertices, a *graph evolution* starts with the empty graph and then randomly adds an edge one at a time until we have reached the complete graph K_n. One of the features of graph evolution is the fact that most property happen suddenly. There is a probability p (or equivalently an expected number of edges m) below which almost no graph has the property but above which almost every graph has this property.

The properties of interest include

(1) The graph has no isolated vertices, the minimal degree $\delta \geq 1$
(2) The graph is no longer a forest, the graph has a cycle
(3) The graph is connected, has finite diameter.
(4) The graph has diameter 2, it is very dense.

Random graph with scilab

Scilab allows one to define functions and the following sequence with generate a matrix that will give us an evolution.

```
function matrix_out = evolution(int_in)
n = int_in;
m1 = rand(n, n); // random matrix
m2 = triu(m1,1); // upper triangular
m3 = m2 + m2'; // make it symmetric ' is transpose
matrix_out = m3;
endfunction
```

The class $e1 = \text{evolution}(5)$ will produced a matrix $e1$ from which an evolution can be obtained via increasing the p and looking at $\text{fix}(e1 + p)$. There are a number of evolution's created for the use in this project. For each roman number r from the list $v(5)$, $x(10)$, $l(50)$, $c(100)$, $d(500)$, and $m(1000)$ there are three graph evolutions with r vertices stored in the downloadable file *evolution.data* (which is in the scilab subdirectory of our homepage. and which can be loaded into the scilab program via the load command. The load command can load some or all these matrices. For example the following command will load the 3 1000×1000 matrices.

```
load('evolution.data', 'm0', 'm1', 'm2');
```

To use many of the built in scilab commands one has to convert the evolution to an adjacency matrix,(via $\text{fix}(e+p)$) and then convert the adjacency matrix to the graph datastructure. The first parameter to mat_2_graph is a sparse matrix, the second parameter of 0 says the graph is not directed, the third parameter says the matrix is an adjacency matrix and not an incident matrix. The following two-liner turns evolution e with $p = 0.345$ winto a scilab graph g

```
p = 0.345;
g = mat_2_graph(sparse(fix(e + p)),0,'node-node');
```

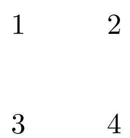
Having the graph g means one can find its diameter with the command graph_diameter(g). Test if g is connected using the command is_connex(g). Find the number of conponents using connex(g) and hence the number of fundemental cycles if you know the number of edges. Unfortunately many of the build-in scilab graph commands expect a directed graph and not our undirected graph. For example hamilton(g) will say it needs a directed g. Of course, changing the second parameter to mat_2_graph from '0' to '1' will produce a directed graph which one can feed to the hamilton command. Be careful, graph properties are often slightly different for directed and undirected graphs.

Evolution example

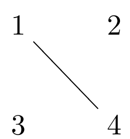
Lets do a simple example with 4 vertices. Assume the evolution matrix is given by

$$A = \begin{bmatrix} 0.0 & 0.5 & 0.2 & 0.7 \\ 0.5 & 0.0 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.0 & 0.6 \\ 0.7 & 0.4 & 0.6 & 0.0 \end{bmatrix}$$

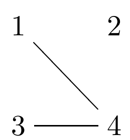
Here are the pictures of the graphs at various points in the evolution. We graph with the adjacency matrix given by fix$(A+p)$ in scilab form. Note the the high entries produce edges first. It is $a_{i,j} + p \geq 1$ and not $a_{i,j} \leq p$ with produces an edge.



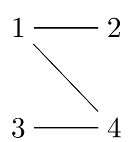
$p = 0.0$ zero edges all isolated vertices



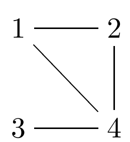
$p = 0.3$ one edge, still a forest



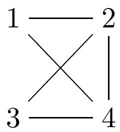
$p = 0.4$ two edges, still a forest



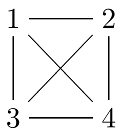
$p = 0.5$ three edges, connected and a tree, no isolated vertices, diameter is 3.



$p = 0.6$ four edges, first cycle, diameter 2.



$p = 0.7$ five edges, two fundamental cycles.



$p = 0.8$ six edges, 3 fundamental cycles, the complete graph K_4

Deliverables

Everyone has a slightly different assignment. How do we get our assignment numbers (0 through 16)? They are in the on-line grade book. Besides the data/plots/questions ask for below, you need to describe the process used to obtain your goal in enough detail so you could repeat it five years from now. Also you need to describe how you know the program is not giving you garbage answers. Let y be your number

(1) A plot of the number edges vs p for the three evolutions, vi, li, di where $i = y \mod 3$. This should be a test of the randomness as we expect this to be close to a straight line. Draw the graph for xi at the probability p just after the last isolated vertex disappears. Your drawing should be good enough to see the number of components and the number of cycles.

(2) For each of the evolutions, find the probability when then last isolated vertex disappears. This can be computed from the evolution directly without making it graph.

(3) Let $z = y/3$ the integer division without remainder. Use a binary search or another method on vi, xi, li, ci, di, mi where $i = z \mod 3$ to compute the onset of diameter becoming 2. you need at least two and preferably 3 significant digits for p. (0.001 only has one significant digit)

(4) Let $w = y/5$ the integer division without remainder. Find the onset of connectivity for the graphs xi, ci, di, mi where $i = w \mod 3$.

(5) Let $x = y/2$ the integer division without remainder. Find the onset of the first cycle for the graphs vi, li, di where $i = x \mod 3$. Can you answer how long is this cycle?

(6) Extra Credit. Find the number of vertices in the largest component when the isolated vertices disappear. (This is the size of the large component.)