1 Separation of Variables

Find the solution \( u(x, y) \) to the following equations by separating variables.

1. \( u_x + u_y = 0 \)
2. \( u_x - u_y = 0 \)
   answer: \( u = ce^{k(x+y)} \)
3. \( y^2u_x - x^2u_y = 0 \)
4. \( u_x + u_y = (x + y)u \)
   answer: \( u = c \exp \left[ \frac{1}{2} (x^2 + y^2) + k(x - y) \right] \)
5. \( u_{xx} + u_{yy} = 0 \)
6. \( u_{xy} - u = 0 \)
   answer: \( u = c \exp(kx + y/k) \)
7. \( u_{xx} - u_{yy} = 0 \)
8. \( xu_{xy} - 2yu = 0 \)
   answer: \( u = x^k e^{-y^2/k} \)

Solution to #4 above. Let \( u = X(x)Y(y) \), plugging to the equation gives

\[
X'(x)Y(y) + X(x)Y'(y) = (x + y)X(x)Y(y)
\]

\[
\frac{X'(x)}{X(x)} + \frac{Y'(y)}{Y(y)} = x + y
\]

\[
\frac{X'(x)}{X(x)} - x = k = y - \frac{Y'(y)}{Y(y)}
\]

for some constant \( k \). We have two ODE to solve

\[
X'(x) - (x + k)X(x) = 0 \quad \text{and} \quad Y'(y) - (y - k)Y(y) = 0
\]

The first has an integrating factor of \( \exp(-x^2/2 - kx) \) and solution \( X(x) = C \exp(x^2/2 + kx) \). The second has an integrating factor of \( \exp(-y^2/2 + ky) \) and solution \( Y(y) = C \exp(y^2/2 - ky) \). Multiplying the ODE solutions gives the answer above.

Solution to #7. \( u = X(x)Y(y) \)

\[
X''(x)Y(y) - X(x)Y''(y) = 0
\]

\[
\frac{X''(x)}{X(x)} = k = \frac{Y''(y)}{Y(y)}
\]
\[ X''(x) - kX(x) = 0 \quad Y''(y) - kY(y) = 0 \]

Supposing \( k \neq 0 \), we get \( X(x) = C_1e^{\omega x} + C_2e^{-\omega x} \) and \( Y(y) = C_1e^{\omega y} + C_2e^{-\omega y} \), where \( \omega \) is the (possibly complex) number so that \( \omega^2 = k \). Our answer has 4 terms

\[ u = A \exp(\omega(x+y)) + B \exp(\omega(x-y)) + C \exp(\omega(y-x)) + D \exp(-\omega(x+y)) \]

If \( k < 0 \) and changing \( \omega \) so that \( k = -\omega^2 \) we have the alternate solution \( X(x) = C_1 \cos \omega x + C_2 \sin \omega y \) and \( Y(y) = C_1 \cos \omega y + C_2 \sin \omega y \) Our answer has four different terms

\[ u = A \cos \omega x \cos \omega y + B \cos \omega x \sin \omega y + C \sin \omega x \cos \omega y + D \sin \omega x \sin \omega y \]

Finally if \( k = 0 \), \( X(x) = C_1 x + C_2 \) and \( Y(y) = C_1 y + C_2 \) giving the solution

\[ u = Axy + Bx + Cy + D \]

### 2 Characteristic examples, Normal form table

If the PDE is \( au_{xx} + bu_{xy} + cu_{yy} = 0 \) and the roots of \( ax^2 - bx + c \) are \( r \) and \( s \). (Note the sign change from \( b \) in the PDE to \( -b \) in the polynomial.) The constant coefficient case looks like:

<table>
<thead>
<tr>
<th>Type</th>
<th>Hyperbolic</th>
<th>Parabolic</th>
<th>Elliptic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Roots ( r ) and ( s )</td>
<td>real and ( r \neq) ( s )</td>
<td>real and ( r =) ( s )</td>
<td>complex ( r = a + bi ), ( s = a - bi )</td>
</tr>
<tr>
<td>Characteristics</td>
<td>( \Phi = y - rx ), ( \Psi = y - sx )</td>
<td>( \Phi = \Psi = y - rx )</td>
<td>( \Phi = y - rx ), ( \Phi = y - sx )</td>
</tr>
<tr>
<td>New variables</td>
<td>( \xi = y - rx ), ( \eta = y - sx )</td>
<td>( \xi = x ), ( \eta = y - rx )</td>
<td>( \xi = y - ax ), ( \eta = bx )</td>
</tr>
<tr>
<td>Solution</td>
<td>( u = f(y - rx) + g(y - sx) )</td>
<td>( u = f(y - rx) + xg(y - rx) )</td>
<td>( u = f(y - rx) + g(y - sx) )</td>
</tr>
<tr>
<td>Normal form</td>
<td>( u\xi_y = 0 ) or ( u\xi - u\eta = 0 )</td>
<td>( u\eta = 0 )</td>
<td>( u\xi + u\eta = 0 )</td>
</tr>
</tbody>
</table>

Some motivation for why this works.

Of course the most interesting question is why the sign change? It is not hard to check that \( ax^2 + bx + c \) and \( ax^2 - bx + c \) have the roots that are negative of each other. So if \( r \) and \( s \) are roots of \( ax^2 - bx + c \) then \( -r \) and \( -s \) are roots of \( ax^2 + bx + c \). Eventually this means \( ax^2 + bx + c = a(x + r)(x + s) \). Symbolically we can write

\[ a \left( \frac{\partial}{\partial x} + r \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} + s \frac{\partial}{\partial y} \right) u = au_{xx} + bu_{xy} + cu_{yy} = 0 \]

If you look at \( u_x + ru_y = 0 \), this says that the directional derivation of \( u \) in the \( (1, r) \) direction is always zero. So \( u \) is constant along lines perpendicular to \( (-r, 1) \), that is \( u \) is constant on lines of the form \( y - rx = C \) for some constant \( C \). This change of sign reflects the change from the direction to the normal direction.

### 3 Characteristic examples, Normal form problems

- We do the wave equation first \( c^2 u_{xx} - u_{yy} = 0 \). Step 1: \( A = c^2 \), \( B = 0 \), \( C = -1 \) and thus \( AC - B^2 = -c^2 < 0 \) so the equation is hyperbolic.

Step 2: is the find the characteristics, we need to solve

\[ A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0 \]

\[ c^2 \left( \frac{dy}{dx} \right)^2 - 1 = 0 \]

\[ \frac{dy}{dx} = \pm 1/c \]

Which gives \( y = x/c + C \) and \( y = -x/c + C \) so \( \Phi = x - cy \) and \( \Psi = x + cy \) are the characters.
Step 3: We solve the equation as \( u = f(x - cy) + g(x + cy) \) Check that it solves the equation.

Step 4: Transforms \( \xi = x - cy \) and \( \eta = x + cy \) gives \( u_x = u_\xi + u_\eta, u_y = -cu_\xi + cu_\eta, \) 
\( u_{xx} = u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta}, u_{yy} = c^2u_{\xi\xi} - c^2u_{\xi\eta} - c^2u_{\eta\xi} + c^2u_{\eta\eta}, \) So
\[
c^2u_{xx} - u_{yy} = 4c^2u_{\xi\eta}
\]
and the equation has the canonical form \( u_{\xi\eta} = 0 \)

- Problem #13 in §12.4 gives the PDE \( u_{xx} + 9u_{yy} \) and asks us to find the type, transform to normal form and solve. Step 1 is to classify the equation, clearly \( A = 1, B = 0 \) and \( C = 9 \) so that \( AC - B^2 = 9 > 0 \) and the equation is elliptic.

Step 2 is to find the characteristics, we need to solve
\[
A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0
\]
\[
\left( \frac{dy}{dx} \right)^2 + 9 = 0
\]
\[
\frac{dy}{dx} = \pm 3i
\]
Which gives \( y = 3ix \) and \( y = -3ix \), we write these as \( \Phi = y - 3ix \) and \( \Psi = y + 3ix \) as characteristics.

Step 3 from the characteristics, we can solve the equation as
\[
u(x, y) = f(y - 3ix) + g(y + 3ix)
\]
Note assuming complex variables behave
\[
u_{xx} = (-3i)^2f''(y - 3ix) + (3i)^2g''(y + 3ix) = -9f'' - 9g''
\]
\[
u_{yy} = f''(y - 3ix) + g''(y + 3ix) = f'' + g''
\]
and clearly \( u_{xx} + 9u_{yy} = 0 \).

Step 4, we use the transformations \( \xi = (\Phi + \Psi)/2 = y \) and \( \eta = (\Phi - \Psi)/2i = 3x \) to change the PDE to the canonical form \( u_{\xi\xi} + u_{\eta\eta} = 0 \). Eventually \( u_{\xi\xi} = u_{yy} \) and \( 9u_{\eta\eta} = u_{xx} \). The change rule was use in step 4.
\[
u_x = u_\xi \xi_x + u_\eta \eta_x = 0u_\xi + 3u_\eta = 3u_\eta
\]
\[
u_{xx} = 3(u_\eta \xi_x + u_\eta \eta_x) = 9u_{\eta\eta}
\]

- Problem #15 \( u_{xx} + 2u_{xy} + u_{yy} = 0 \) Step 1 \( A = B = C = 1 \), so that \( AC - B^2 = 0 \) and the equation is parabolic.

Step2:
\[
A \left( \frac{dy}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0
\]
\[
\left( \frac{dy}{dx} \right)^2 - 2\frac{dy}{dx} + 1 = 0
\]
factors to \((\frac{dy}{dx}) - 1)^2 = 0\) and there is the one solution \( y = x + C \) so \( \Phi = (y - x) \) is a characteristic

Step 3: We need two equations, the second is \( x \) times something similar to the first so \( u = f(y - x) + xg(y - x) \) (An early verion of this handout had \( f(y - x) + Cx \) which is also a solution by not as general as possible. Then we had \( f(y - x) + xf(y - x) \), which is inbetween, but still not as general as the current answer) Lets check it \( u_x = -f'(y - x) + g(y - x) - xg'(y - x), \) \( u_y = f'(y - x) + 2g'(y - x), \)
Step 2: since the second term is lower order we are ok.

Step 4: Let \( \xi = y - x \) and \( \eta = x \) then \( u_x = -u_\xi + u_\eta \), \( u_y = u_\xi + 0u_\eta \),
\[
\begin{align*}
  u_{xx} &= -(-u_\xi + u_\eta) + (-u_\xi + u_\eta) = u_\xi - 2u_\xi + u_\eta \\
  u_{xy} &= -(u_\xi + 0u_\eta) + (u_\xi + 0u_\eta) = -u_\xi + u_\xi \\
  u_{yy} &= u_\xi + 0u_\eta = u_\xi \\
  u_{xx} + 2u_{xy} + u_{yy} &= (f''(y-x)-2g'(y-x)+g'(y-x))+2(-f''(y-x)+g'(y-x)-g''(y-x))+f''(y-x)+g''(y-x) = 0
\end{align*}
\]

Step 4: Let \( \eta = y - x \) and \( \xi = x \) then \( u_x = -u_\xi + u_\eta \), \( u_y = u_\xi + 0u_\eta \),
\[
\begin{align*}
  u_{xx} &= -(u_\xi + 0u_\eta) + (u_\xi + 0u_\eta) = -u_\xi + u_\xi \\
  u_{xy} &= -(u_\xi + 0u_\eta) + (u_\xi + 0u_\eta) = -u_\xi + u_\xi \\
  u_{yy} &= u_\xi + 0u_\eta = u_\xi \\
  u_{xx} + 2u_{xy} + u_{yy} &= (1-2+1)u_\xi + 2(-1+1+0)u_\xi + (1+0+0)u_\eta = u_\eta \\
\end{align*}
\]
And so the canonical form is \( u_\eta = 0 \).

- Problem #19 Requires more steps than are in the text. It gives the PDE \( xu_{xx} - yu_{xy} = 0 \). Step 1 has \( A = x \), \( B = -y/2 \) and \( C = 0 \), so that \( AC - B^2 = -y^2/4 < 0 \) (if \( y \neq 0 \)) and the equation is hyperbolic.

Step 2:
\[
A \left( \frac{du}{dx} \right)^2 - 2B \frac{dy}{dx} + C = 0
\]
\[
x \left( \frac{dy}{dx} \right)^2 + y \frac{dy}{dx} = 0
\]

This factors into
\[
\frac{dy}{dx} \left( x \frac{dy}{dx} + y \right) = 0
\]
The first ODE is \( \frac{dy}{dx} = 0 \) or \( y = C \) so \( \Phi = y \), the second ODE is \( \frac{dy}{y} = -\frac{dx}{x} \) or \( y = C/x \) or \( xy = C \) so \( \Psi = xy \).

The method of the textbook does not correctly handle the next part of the problem. The method of
textbook does work if \( A, B, C \) are constants. The additional work needed to solve this in this version
of extra.

Step 3: The table in the text implies \( u = f(y) + g(xy) \) should be the solution. But it is not; checking
we see that
\[
\begin{align*}
  u_x &= yg'(xy); \\
  u_{xx} &= y^2g''(xy); \\
  u_{xy} &= xyg''(xy) + g'(xy) \\
  xu_{xx} - yu_{xy} &= xy^2g''(xy) - xy^2g''(xy) - yg'(xy) = 0
\end{align*}
\]
Instead we need another trick.

The trick is to let \( p(x, y) = u_x \), our PDE becomes \( xp_x - yp_y \) which is a first order equation and which
has the general solution \( p = g(xy) \) found above. (This is easy to check.) Now we just solve \( u_x = g(xy) \)
by integration obtaining
\[
u = f(y) + \int g(xy) \, dx = f(y) + h(xy)/y
\]
Why is the \( \int g(xy) \, dx = h(xy)/y \)? Well it has to be something whose \( x \)-partial is a function of \( y \).
So in must be an arbitrary function \( h(xy) \) but we need to make its \( x \)-partial, \( yh(xy) \), be an function of
\( xy \); clearly dividing by \( y \) does the trick. Checking this solution gives
\[
\begin{align*}
  u_x &= yh'(xy)/y; \\
  u_{xx} &= yh''(xy); \\
  u_{xy} &= xh''(xy) \\
  xu_{xx} - yu_{xy} &= xyh''(xy) - xyh''(xy) = 0
\end{align*}
\]
Step 4: \( \xi = y, \eta = x \) \( u_x = 0u_\xi + yu_\eta, u_y = u_\xi + xu_\eta, u_{xx} = y(0u_\xi + yu_\eta) = y^2u_\eta, u_{xy} = u_\eta + y(xu_\xi + u_\eta) = yu_\eta + xyu_\eta + u_\eta, u_{yy} = u_\xi + xu_\xi + x(u_\xi + xu_\eta) = u_\xi + 2xu_\xi + x^2u_\eta
\]
\[
xu_{xx} - yu_{xy} = y^2u_\eta - (y^2u_\eta + xy^2u_\eta + yu_\eta = xy^2u_\eta + yu_\eta
\]
Dividing by \( xy^2 = y\eta \) we get the canonical
\[
u_\eta + u_\eta/\eta = 0
\]
since the second term is lower order we are ok.