Charactizing the Reals and Hyperreals

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1 Definitions

The goal of this handout is the characterize the reals and put them in context with similar structures. The reals $\mathbb{R}$, are a *complete ordered field*, in fact it is the only complete ordered field in some sense. In another sense, the strictly bigger hyperreal field $\mathbb{R}^*$ is also a comple ordered field. But that is jumping the gun.

First we recall the definition of a *field*. Basically a field allows you to add, substract, multiply and divide by non-zero elements as you can do real numbers. Examples of fields include: $\mathbb{Z}_p$, the integers mod p for some prime p, $\mathbb{Q}$, the rationals, $\mathbb{C}$, the complex numbers, and $\mathbb{R}^*$ the hyperreals.

We list the axioms of a field next. A field F has two binary operations $+$ and $\cdot$ so that for all $a, b, c \in F$

1. (Closure) $a + b \in F$ and $a \cdot b \in F$.
2. (Associativity) $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
3. (Commutativity) $a + b = b + a$ and $a \cdot b = b \cdot a$.
4. (Identity) There are $0, 1 \in F$ so that $a + 0 = a$ and $a \cdot 1 = a$.
5. (Inverse) There are $-a$ and, if $a \neq 0$, a^{-1} so that $a + (-a) = 0$ and $a \cdot a^{-1} = 1$.
6. (Distributivity) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Next we add some order axioms, this will eliminate the integers mod p and the complex numbers from the list, but both the rationals and the hyperreals are ordered fields.

An *ordered field* is a field with a binary relation $\leq$, less than, with these axioms, for all $a, b, c \in F$

1. (Reflexivity) $a \leq a$.
2. (Anti-Symmetric) $a \leq b$ and $b \leq a$ implies $a = b$.
3. (Transitivity) $a \leq b$ and $b \leq c$ implies $a \leq c$.
4. (Trichotomy) Either $a = b$ or exactly one of $a \leq b$ or $b \leq a$ is true.

5. (Addition is order preserving) If $a \leq b$, then $a + c \leq b + c$

6. (Multiplication by positives is order preserving) If $a \leq b$, and $0 \leq c$, then $a \cdot c \leq b \cdot c$

A field F is said to be *characteristic* n if n is the smallest integer so that the sum of n-copies of 1 equals 0, or the characteristic is 0 if this never happens. $\mathbb{Z}_p$ has characteristic p and ordered fields have characteristic 0. (See the exercises at the end.)

The next term, completeness, is over used in mathematics. While you might think that we are talking about completeness in the sense that every Cauchy sequence converges, we are not. The property here is also called *order completeness*. An element $a \in F$ is an upper bound (resp. lower bound) for $S \subset F$ if $s \in S$ implies $s \leq a$ (resp $s \in S$ implies $a \leq s$.

A *complete ordered field* is an order field F with the addition property

1. (Existence of sup) each subset $S \subset F$ with an upper bound, has a least upper bound $\sup S$

2. (Existence of inf) each subset $S \subset F$ with an lower bound, has a greatest lower bound $\inf S$

A pair of sets S and T are called a cut of the ordered field F if S and T are a partition of F and $s \in S, t \in T$, implies $s \leq t$. A partition means every element of F is in exactly one of S or T. Equivalently $F = S \cup T$ and $S \cap T = \varnothing$. A cut of the rationals so that S has no largest element is called a Dedekind cut. Each Dedekind cut, (S, T) corresponses to a unique real number r so that $S = \{q \in \mathbb{Q} : q < r\}$ and $T = \{q \in \mathbb{Q} : r \leq q\}$.

2 Exercises

1. If F has characteristic $n > 0$ and $a \in F$, then the sum of n a's is zero.
2. If F has characteristic $n > 0$, then n is prime.
3. If F is an ordered field, then F has characteristic 0.
4. If F has characteristic $p > 0$, then the smallest subfield of F is isomorphic to $\mathbb{Z}_p$.
5. If F has characteristic 0, then the smallest subfield of F is isomorphic to $\mathbb{Q}$.
6. If F is an ordered field, then

   $$a > 0 \iff -a < 0, -a > 0 \iff a < 0, a > 0 \iff a^{-1} > 0, a < 0 \iff a^{-1} < 0.$$

7. If $a \neq 0$ is an element of F, an ordered field, then $a^2 > 0$.
8. In an ordered field: $1 > 0$.
9. $\mathbb{C}$ cannot be an ordered field.
10. $S = \{q \in \mathbb{Q} : q < \sqrt{2}\}$ has 2 as a upper bound but has no $\sup S$ in $\mathbb{Q}$. So $\mathbb{Q}$ is not a complete ordered field.
11. For any cut S and T, if either $\sup S$ or $\inf T$ exist, then so does the other an $\sup S = \inf T$.
12. Any complete ordered field contains $\mathbb{R}$ as a subfield.
13. Any ordered field F that is strictly bigger than $\mathbb{R}$ contains non-zero infinitesimals and infinite numbers.
14. If $\mu(0)$ has more than one element, then it is a set with an upper bound but no least upper bound.
15. For any cut S and T and any $\epsilon > 0$, there are $s_\epsilon \in S, t_\epsilon \in T$ with $|t_\epsilon - s_\epsilon| < \epsilon$
16. For any cut S and T and integer n, there are $|t_n - s_n| < 1/n$ with $s_n \in S, t_n in T$ and with

    $$s_1 < \ldots < s_n < s_{n+1} < \ldots < t_{n+1} < t_n < \ldots t_1$$

17. If the sequence $\{s_n\}$ above converges to s_∞ then, $s_\infty = \sup S = \inf T$ and is the limit of $\{t_n\}$.
18. An ordered field F is Archimedean if for each $a \in F$ there is an integer n so that $a \leq n$. Show this is equivalent to each $a > 0$ there is an integer n so that $a > 1/n$.

19. The reals $\mathbb{R}$ and the rationals $\mathbb{Q}$ are Archimedean.
20. An Archimedean ordered field has no infinite elements.
21. An Archimedean ordered field has no non-zero infinitesimal elements.
22. An ordered field with no infinite elements is Archimedean