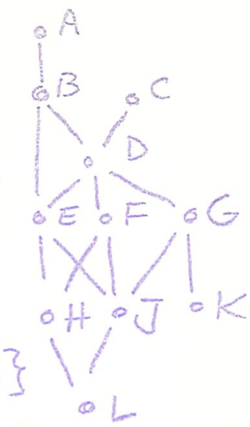
DM 2 test 1 SHOW ALL WORK #1-4: 10pts NAME ______
#5-8: 15pts NUMBER ______

1. For the p.o. "graph" to the right
A. List all maximal elements. ______
B. List all minimal elements. ______
C. List all upper bounds to the set $\{H, J, K\}$ ______
D. Find lub $\{H, G\}$ ______
E. Find glb $\{A, D\}$ ______



2. The relation R is defined on the set $\{1, 2, 3, \dots\}$ by $xRy \iff x+2 \le y$. For each property below say "yes" if R has that property else say "no" and give a counter example

A. Reflexive	B. Transitive	C Symmetric	D. Anti-symmetric

3. DRAW Digraphs showing relations with the required properties
A. A relation which isn't transitive

B. A relation which is neither symmetric nor anti-symmetric

C. An equivalence relation on 3 or more vertices which isn't the relation "=" and has more than one equivalence class.

4. A recurrence relation has characteristic polynomial $(x^2+1)(x-1)(x+3)^2$ A. Write the general solution to the homo problem

BCDE For the given forcing fcn f(n), write the correct guess for the particular solution
B. $8 \cdot 2^n$ ______ C. $100n$ ______ D. $6i^n$ ______ E. $(n+7)3^n$ ______

5. Solve $a_n + 2a_{n-1} = 10 \cdot 3^n$ $a_0 = 20$.

6\. A. Find a pair of elements in the p.o. graph in Prob 1 which do not have a lub!

B. Find a counterexample to "a p.o. set with a least element is well-ordered"

C. Find a counter example "if R & S are irreflexive, then RS is irreflexive"

D. Find two positive integers x, y so that lcm (x, y) isn't x, y or xy.

E. Subsets A & E so that $(A-E)\cup E \neq E$

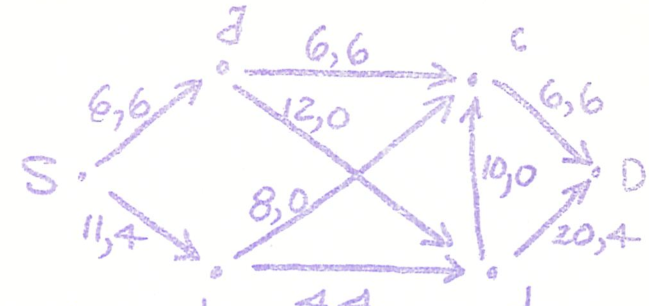
7\. Given $a_1 = 5$ and $a_n = 9 - 2(1-\frac{1}{n})a_{n-1}$ for $n \geq 2$
Prove by induction $a_n = 3 + \frac{2}{n}$ for $n \geq 1$.

8\. Prove: if the finite p.o set A has exactly one minimal element s, then s is the least element of A. (You may use the fact that each finite p.o. set has a minimal element).

DM2 Test 2 Show ALL work for credit By ______

1 – 4 are worth 10 pts each, 5–8 15 pts each. No ______

1\. A. Find a F-augmenting path for the flow F on the transport network to right



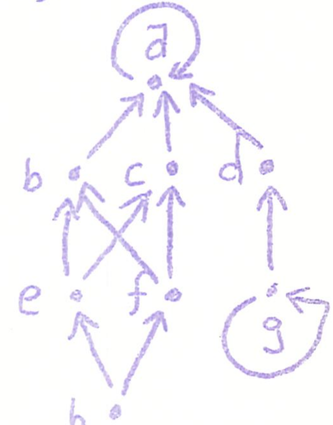
B. On the transport network show how ~~to~~ increase the flow using your path in A.

2\. The matrix ($k=0$) is an adjacency matrix of a digraph G. Use Warshall's algorithm to fill in the other matrices

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$k=0$ $\qquad k=1 \qquad k=2 \qquad k=3 \qquad k=4$

3\. A. Construct the adjacency matrix for the digraph to the right



B. Give any topological enumeration of this digraph

C. Give another topological enumeration different from the one given in B

4\. Solve $a_n - 5a_{n-1} + 6a_{n-2} = 0$ $\quad a_0 = 1 \quad a_1 = 0$.

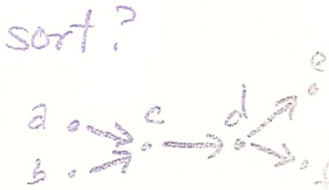
5\. A. give an example of a maximal directed path which isn't a directed path of maximal length.

B. draw a digraph of a non-reflexive relation whose transitive closure is reflexive

C. give a permutation of $\langle 1,2,3,4,5\rangle$ which has a disorder of 3

D. What is runtime complexity of interchange sort?

E. List ALL the unilaterally connected components of



6. Give Counterexamples:

A. A flow F on a transport network with zero flow on each out of S must have zero flow on every edge.

B. A graph with two distinct undirected paths x to y must have a undirected cycle.

C. If each cycle in G has odd length, then each circuit in G has odd length.

D. If each vertex v in G has in deg(v) = out deg(v) = 1 then G is strongly connected

E. An F augmenting path P for a flow F must be a simple path

7. A. Rewrite $3^{\log_9 n}$ as n^k with k as simple as possible

BCDE. Solve $a_n = 2a_{n/3} + n \quad a_1 = 4$

8. Prove: If G is a digraph (V, E) and $n \geq 1$, then $(x,y) \in E^n \Longleftrightarrow$ (if and only if) there is a directed path of length n from x to y.

Final DM2 Fall 86

Show ALL Work for credit.

1-8; 10 pts; 9-16 15pts.

name ______________________

SS# ______________________

1.A. Find a F-augmenting path for the flow F on the transport network in figure 1.

B. On the transport network, show how to increase the flow using your path in part A.

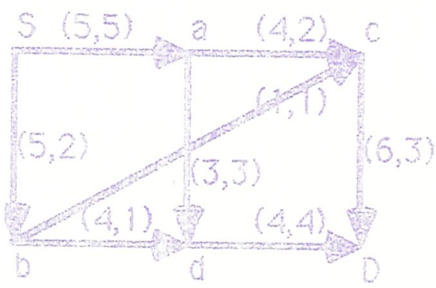


fig. 1

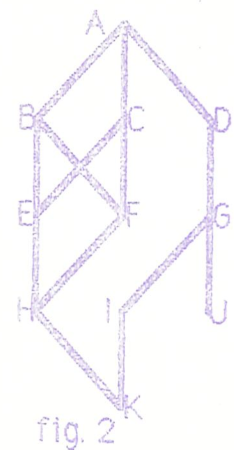


fig. 2

2.For the partial order "Hasse" graph in figure 2 answer:

A. The maximal elements are: ______________________

B. The minimal elements are: ______________________

C. Find glb {F, I}: ______________________

D. Give a topological enumeration: ______________________

E. Find lub {E, F}: ______________________

3. The matrix (k=0) is an adjacency matrix of a digraph G. Use Warshall's algorithm to fill in the other matrices.

$$\begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 1&0&0&1 \\ 0&0&1&0 \end{bmatrix} \quad \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \quad \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \quad \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \quad \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

k = 0 k = 1 k = 2 k = 3 k = 4

4. Define the relation R on the set of non-zero rational numbers by $xRy \Leftrightarrow x/y$ is an integer. For each of the property below say "yes" if R has that property else say "no" and give a counterexample.

reflexive	transitive	symmetric	anti-symmetric

5.A.Company XYZ has 200 ditch diggers, 110 have round bottom shovels, 37 have both round and square bottom shovels and 22 don't have shovels. How many have square bottom shovels?

B. How many ways are there to put 202 people into 4 rooms with at least one person in each room?

6. Solve $a_n = a_{n-1} + 6a_{n-2}$; $a_0 = 1$, $a_1 = 0$.

7. A recurrence relation has characteristic polynomial $(x^2+1)(x-1)(x-2)(x+3)^3$
A. Write the general solution to the homogenuous problem.

BCDE. For the given forcing function, write the correct quess for the form of the particular solution:
B. $8 \cdot 4^n$ ________ C. $10n$ ________ D. $6i^n$ ________ E. $n(-3)^n$ ________.

8. Solve $a_n = 3a_{n/7} + 2n$; $a_1 = 5$;

9.A.Find the coefficient of x^{40} in:

$$(1 + x + x^2 + \ldots)^5 (x^5 + x^6 + \ldots + x^9)^3 (x^3 + x^4 + x^5 + \ldots)^4$$

B.Write (but do NOT solve) a generating function and tell which coefficient we need to find in order to count the number of ways of putting 50 identical balls into 20 boxes so that odd numbered boxes have either 1, 3 or 7 balls, boxes 2,4,6,8 and 10 each have at least 2 balls and the rest of the boxes have no more than 3 balls.

10. Solve by generating functions: $a_0 = 16$; and for $n>0$, $a_n = 3a_{n-1} + 4 \cdot 5^n$.

11. Give examples:
A. A circuit which isn't a cycle.

B. A digraph of a relation which isn't reflexive or irreflexive.

C. A digraph of a symmetric relation which isn't transitive.

D. Find all the unilaterally connected components of

E. Give the run time complexity of a binary search in an array of length n.

12 & 13. Give counterexamples:
A. A non-trival closed path is a circuit.

B. A maximal matching which isn't a complete matching.

C. A maximal directed path is a directed path of maximal length.

D. A transport network with a unique minimal cut has a unique maximal flow.

E. A transport network with a unique maximal flow has a unique minimal cut.

F. A partial order on more than 2 elements is never an equivalence relation.

G. For each choice of X being one of the words "weakly", "unilaterally" or "strongly", each X connected digraph G, has a vertex v, so that G-v is X connected.

H. If F is a maximal flow and $k(X, X^c) = F(X, X^c)$, then (X, X^c) is a minimal S-D cut.

I. A partially ordered set with only one maximal element has a maximum element.

J. The lexicographic ordering on strings of the usual 26 letters is a well ordering.

14. Prove by induction:

Given $a_n = \sqrt{6 + a_{n-1}}$ for $n \geq 1$, and $a_0 = \sqrt{3}$. Prove $a_n < 3$, for $n \geq 0$.

15. Prove a partial order is a total order if and only if its digraph is unilaterally connected.

16. Prove: If $G = (V, E)$ is a digraph and $n > 0$, then (x, y) is in E^n if and only if there is a directed path of length n from x to y in G.

DM2 PT2 Some Old Problems on old tests:

1. A $4^{\log_3 n} = n^k$ so $k = ?$
(15pts) BCDE Solve $a_n = 2a_{n/3} + 4$, $a_1 = 5$

2. (10pts) Define A. $k(X, \bar{X})$ (on a network) B. A circuit

3. (10pts) Give counter examples: A. (5pts) If E is an edge $x \to y$ in the connected graph G ~~so that each vertex of G is~~ and E is a cut-edge of G then the only path $x \to y$ in G is E ~~B. If v, w~~

B. (4pts) If P, Q $x \to y$ are different simple paths then P followed by Q backwards is a circuit.

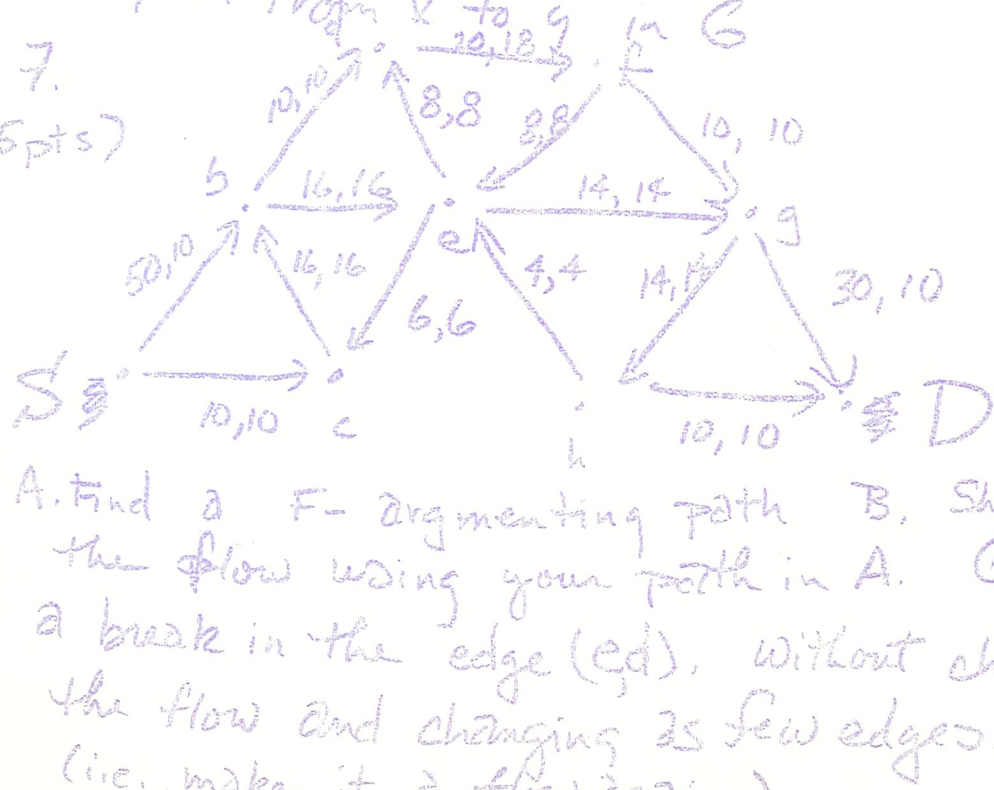
C (4pts) A connected graph with 6 vertices and 7 edges has exactly two circuits (Indicate the circuits)

4. (15pts) Solve $a_n = 3a_{n/8} + 2n$, $a_1 = 1$

5 (15pts) G is a conn graph and E is an edge in G. Prove E is a cut-edge iff E is no circuits of G

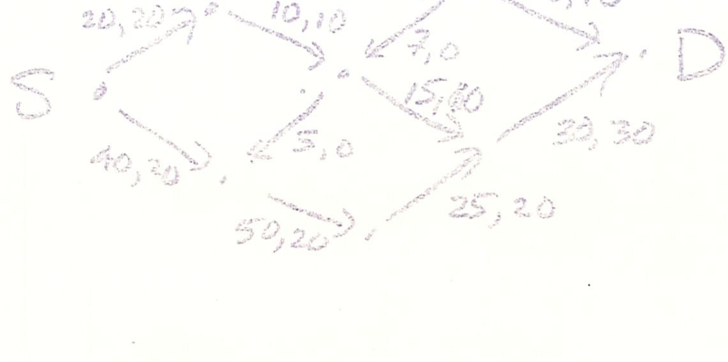
6 (15pts) Prove by induction (on the crossing no.) If there is a path from x to y in G, then there is a simple path from x to y in G

7. (15pts)



A. Find a F-augmenting path B. Show how to increase the flow using your path in A. C. Suddenly there is a break in the edge (e,d). Without changing the value of the flow and changing as few edges as poss, correct the (i.e. make it a flow again.)

8 (10pts) ditto for



Pg 2 M272 Some new questions

1. Give Counterexamples: Each circuit is a cycle. Each simple closed path is a cycle. Each closed path with at least one edge is a circuit. An F-augmenting path can have no saturated edges. If $k(X, \bar{X})$ is a min cut and F is a flow so that each edge from a vertex in X to a vertex in $\bar{X}$ is saturated, then F is a max flow. Interchanging a_i and a_j decreases the disorder of the array $a_1, a_2, \ldots, a_5$ when $1 \le i < j \le 5$. Merging two list with 7 elements each takes at most 8 comparisons, at least 8 comparisons

2. Solve $a_n - 3a_{n-1} + 6a_{n-2} = 0 \quad a_1 = a_2 = 10$

3. Illustrate: Merge sort on the list $< 8\ 1\ 4\ 3\ 5\ 6\ 2\ 7 >$

Binsearch (25) on $< 5, 10, 15, 20, 25, 30, 35 >$

Topsort on G below, Warshall's Algorithm on G below

(Chose the vertex with smaller lexicographic order when poss.)

4. What is the run time complexity of Warshalls, Top sort, merge sort, binsearch?

5. Give examples: A unilaterally connected graph which has 3 strongly connected components. A weakly connected graph which isn't unilaterally connected. Two ~~strongly~~ unilaterally connected components which have a non-void intersection. An edge which is in no strongly conn. comp.

6. Prove by induction $a_n = 2^{(3^n + (-1)^n)}$ is the solution to $a_n = (a_{n-1})^2 \cdot (a_{n-2})^3 \quad a_0 = a_1 = 4$ [hint take logs]

7. Prove a digraph G is strongly connected if and only if G contains a closed path with all the vertices of G

DM2 Practice Test 3 3 DEC 1986

Part I: From old tests 1 - 7 were 10 points, 8 - 9 were 15 points.

1. Write (but do <u>not</u> solve) a generating function and tell which coefficient we need to find the number of non-negative integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 27$$

with $0 \le x_1, x_2, x_3$; $3 \le x_4 \le 7$ and x_5 is an even integer between 1 and 7.

2. Find the coefficient of x^{24} in $(x^3 + x^4 + x^5 + x^6 + x^7)^4 (1 + x + \dots)^{12}$.

3. Find the generating function with $a_r = 4r^2$.

4. Solve $a_n = 4a_{n-1} + 6^n$ (for $n \ge 1$), $a_0 = 2$ by generating functions.

5. Use generating functions to find how many ways there are to put 30 identical balls into 9 distinct boxes so that the first three boxes have at least two balls each and the last three boxes have no more than 10 balls each.

6. Use Inclusion Exclusion to find the number of ways in which you can pick 20 electronic games from Santa's 50 kinds if you can pick up to 5 games of the same kind.

7. How many ways are there to roll 100 distinct dice with at least one of each kind of face (i.e. each of the numbers 1 through 6) showing?

8. Solve by GENERATING FUNCTIONS $a_n = 5a_{n-1} + 2^n$ (for $n \ge 1$); $a_0 = 2$.

9. A. Find the number of arrangements of ALGORITHM with the A before the L or the L before the G or the G before the A.

B. Find the number of 7 card poker hands with at least one card in each suit.

Part II: More questions

10. Solve for a_n given

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{30 - 2x^2}{(1-2x)(1+3x)(1-10x)}$$

11. Given $|A| = 100$, $|B| = 112$, $|C| = 95$, $|A \cap B| = 50$, $|A \cap C| = 47$, $|B \cap C| = 35$, and $|A \cap B \cap C| = 13$; find $|A \cup B \cup C|$.

12. Give Counterexamples:

A. If a flow F and a S-D cut (X, X^c) on a transport network satisfy $F(X, X^c) = k(X, X^c)$ then F is a maximal flow.

B. A maximal matching is a complete matching.

C. If G is a directed bipartite graph with $|A| \le |B|$, then a maximal matching is a complete matching.

D. A transport network with only one minimal cut has a unique maximal flow.

E. A transport network with a unique maximal flow has a unique minimal cut.

13. Suppose (G,k) is a transport network with integer edge capacities. Suppose further that $e_1 = (v_0, v_1)$, $e_2 = (v_1, v_2)$, ..., $e_k = (v_{k-1}, v_0)$ is a directed k-cycle where $k(e_i) > 0$ for $1 \le i \le k$. Prove or disprove: there is a maximal flow F on G with the property that $F(e_i) > 0$ for all $1 \le i \le k$.

14. Prove:

$$\binom{n+k}{n-1} = \sum_{i=0}^{n-1} \binom{i+k-1}{i}$$

Hint: Count the non-negative integer solutions to $x_1 + \dots + x_k < n$ in two ways.

good luck!

TP's

(note these rules do NOT apply to HW)

1. Rules
   A. They must be on $8\frac{1}{2} \times 11$ paper
   B. They must be written in ink
   C. They must use one side of each page
   D. If there is more than one page, then the pages must be stapled or paper-clipped together.

Failure to follow any rule cost a point each

2. Grades
   A. Graded on a 0 to 10 basis
   B. Graded on your reasoning, your ability to express your reasoning, neatness and your English.

3. TP average is computed using only your best n out of the m assigned where $\frac{n}{m} \geq \frac{2}{3}$

4. Since TP's are assigned a week in advance of their due date, the solutions are graded as if they are carefully worked out.

5. They must be your OWN work

TP 1 due 3 sept 86 Wed

Given: $a_0 = 10$ and for $n \geq 1$ $a_n = 3a_{n-1} + 2 \cdot 3^n$

Prove by induction $a_n = 10 \cdot 3^n + 2n3^n$

Let $P, \leq$ be a poset and let $B \subset P$

1. An element $b \in B$ is called the <u>least</u> or <u>minimum</u> element (resp. <u>greatest</u> or <u>maximum element</u>) of B if for $x \in B$ then $x \geq b$ (resp $x \leq b$)

<u>Thm:</u> B can have at most one least element (or greatest element)

2. An element $b \in B$ is <u>minimal</u> (resp <u>maximal</u>) if $x \in B$ and $x \leq b \Rightarrow x = b$ (resp $x \in B$ & $x \geq b \Rightarrow x = b$)

" A maximum element is maximal but not conversely

3. An element $a \in P$ is an <u>upper bound</u> for B if $b \in B \Rightarrow a \geq b$

An element $c \in P$ is an <u>lower bound</u> for B if $b \in B \Rightarrow c \leq b$.

4. If the set of lower bounds of B has a greatest element, that element is called the <u>greatest lower bound</u> or glb (or inf) similarly B may have a <u>least upper bound</u> (lub or sup)

5. A <u>lattice</u> is a poset in which each pair of elements has a least upper bound and a greatest lower bound.

TP 6 due 15 sept

A. Prove by induction Each finite poset has a maximal element.

B. Each finite lattice has a maximum element.

TP due mon 29 sept

complete the <u>following</u> <u>proof</u> of the theorem

Each finite p.o. set has a maximal element

<u>proof</u>: By induction on the number of elements in the p.o. set.

<u>start-up</u>: Suppose A is any p.o. set ordered by $\leq$ and A has one element b.

Now b is maximal since "$x \in A \Rightarrow x = b$" is true so that "$x \in A$ and $x \geq b \Rightarrow x = b$" is certainly true.

<u>inductive step</u>

we assume each p.o. set with n or fewer elements has a maximal element.

Now let $A, \leq$ be a p.o. set with n+1 elements.

you
fill in
this part

$\therefore$ A has a maximal element $\square$

Solve $a_n = 2a_{n-1} + 3^n \quad (n \geq 1)^{*} \quad a_0 = 5$ by <u>generating</u> <u>functions</u> [we know answer is $a_n = 2 \cdot 2^n + 3 \cdot 3^n$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ (the generating fun!

since $(a_0 + a_1 + a_2 + \dots) = a_0 + (a_1 + a_2 + \dots)$ we need to do this because of *

$$= a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n$$

since $a_0 = 5$ is given & $a_n = 2a_{n-1} + 3^n$ is given for $n \geq 1$

$$= 5 + \sum_{n=1}^{\infty} (2a_{n-1} + 3^n) x^n$$

distributive & commutative law

$$= 5 + \sum_{n=1}^{\infty} 2a_{n-1} x^n + \sum_{n=1}^{\infty} 3^n x^n$$

factor 2x out of 1st sum. Use $(b_1 + b_2 + \dots) = (b_0 + b_1 + \dots) - b_0$ on 2nd sum

$$= 5 + 2x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=0}^{\infty} 3^n x^n - 3^0 x^0$$

$5 - 1 = 4$; $(a_{1-1} x^{1-1} + a_{2-1} x^{2-1} + \dots) = (a_0 x^0 + a_1 x^1 + \dots)$; $\frac{1}{1-3x} = \sum_{n=0}^{\infty} 3^n x^n$

$$= 4 + 2x \sum_{n=0}^{\infty} a_n x^n + \frac{1}{1-3x}$$

$4 = \frac{4 - 12x}{1-3x}$; $A(x) = \sum_{n=0}^{\infty} a_n x^n$ subst.

$$= 2x\, A(x) + \cancel{\frac{4+6x}{1-3x}} \quad \frac{5-12x}{1-3x}$$

or we can re write this as

$$A(x)(1-2x) = \frac{5-12x}{1-3x}$$

or via partial fractions

$$A(x) = \frac{5-12x}{(1-2x)(1-3x)} = \frac{C}{1-2x} + \frac{D}{1-3x}$$

we obtain [multiply by $(1-2x)$ & subst $\frac{1}{2}$] $C = 2$ [multiply by $(1-3x)$ & $x = \frac{1}{3}$] $D = 3$

$$\sum_{n=0}^{\infty} a_n x^n = A(x) = \frac{2}{1-2x} + \frac{3}{1-3x} = 2\sum_{n=0}^{\infty} 2^n x^n + 3 \sum_{n=0}^{\infty} 3^n x^n$$

or

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (2 \cdot 2^n + 3 \cdot 3^n) x^n$$

$$\therefore \quad a_n = 2 \cdot 2^n + 3 \cdot 3^n$$

$$\frac{1}{1-bx} = \sum_{n=0}^{\infty} b^n x^n \qquad \frac{1}{1+bx} = \sum_{n=0}^{\infty} (-1)^n b^n x^n$$

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