We have shown how to take $f \in L_1(\mathbb{R})$, $f \geq 0$ and use it to construct μ a Borel measure with density f. We have shown that $\mu << m$ and that if F_μ is the distribution function of μ then $F_\mu' = f$ a.e. Thus if we know that μ has a density function we can construct it.

The next problem we will consider is how can you tell (from μ alone) if μ has a density function? We will solve this problem by looking at F_μ, and the answer has to do with absolute continuity. The following definition is needed.

<u>DN 1</u>. An increasing function $G: \mathbb{R} \to \mathbb{R}$ is said to be <u>absolutely continuous</u> on $[a,b]$ if for each $\varepsilon > 0$ there is a $\delta > 0$ so that for each integer n and pts $x_1 \cdots x_n$ & $y_1, \ldots, y_n$ satisfying

(1) $a \leq x_1 \leq y_1 < x_2 \leq y_2 < x_3 \leq y_3 < \cdots < x_n \leq y_n \leq b$

(2) $\sum_{i=1}^{n} (y_i - x_i) < \delta$

then we have $\sum_{i=1}^{n} [G(y_i) - G(x_i)] < \varepsilon$.

<u>LEMMA 2</u> If the increasing function G is absolutely continuous on $[a,b]$ then G is continuous on $[a,b]$.

<u>Note:</u> The converse of Lemma 2 is False!

<u>pf of Lemma 2:</u> Suppose for some $c \in [a,b]$, G is not

continuous at c then $G(c^+) > G(c^-)$. Let $\varepsilon = \frac{1}{2}(G(c^+) - G(c^-))$ then for each $\delta > 0$ $x_1 = c - \delta/3$, $y_1 = c + \delta/3$ satisfies (1) and (2) of DN1 but $G(y_1) - G(x_1) \geq G(c^+) - G(c^-) > \varepsilon$. Thus G is not absolutely continuous.

Proposition 3: The Borel measure μ is absolute continuous, if and only if F_μ is absolutely continuous on $(-\infty, \infty)$.

proof: $(\Rightarrow)$ If $\mu \ll m$, then F_μ is continuous, and for each $\varepsilon > 0$ $\exists \delta > 0$ so that for $E \in \mathcal{B}$, $m(E) < \delta \Rightarrow \mu(E) < \varepsilon$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be the resulting δ in $\mu \ll m$. Condition (1) & (2) can be restated as $m(\bigcup_{i=1}^n (x_i, y_i]) < \delta$ and $\sum_{i=1}^n [F_\mu(y_i) - F_\mu(x_i)] = \sum_{i=1}^n \mu((x_i, y_i]) = \mu(\bigcup_{i=1}^n (x_i, y_i]) < \varepsilon$.

$(\Leftarrow)$ If F_μ is absolute continuous and let $E \in \mathcal{B}$ so that $m(E) = 0$; we want to show $\mu(E) = 0$, we will show that for each $\varepsilon > 0$, $\mu(E) \leq \varepsilon$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be the resulting δ from DN1. There are (a_n, b_n), pairwise disjoint open intervals so that $E \subset \bigcup_{n=1}^\infty (a_n, b_n)$ and $\sum_{n=1}^\infty (b_n - a_n) < \delta$. Thus for each m, $\mu(\bigcup_{n=1}^m (a_n, b_n]) = \sum_{n=1}^m \mu((a_n, b_n]) = \sum_{n=1}^m (F_\mu(b_n) - F_\mu(a_n)) < \varepsilon$ [note that $\{(a_n, b_n)\}_{n=1}^m$ may have to be re-ordered to satisfy DN1, but this will not effect the sum]. Therefore $\mu(E) \leq \mu(\bigcup_{m=1}^\infty (\bigcup_{n=1}^m (a_n, b_n]) = \lim_{m \to \infty} \mu(\bigcup_{n=1}^m (a_n, b_n]) \leq \varepsilon$, which completes the proof.

Now let us consider a general Borel measure μ. We know that if $f = F_\mu'$ then $f \geq 0$ and $f \in L_1(\mathbb{R})$. Thus we can define a Borel measure ν with density f.

Lemma 4: With ν, μ as above, then $\forall E \in \mathcal{B}$ $\nu(E) \leq \mu(E)$.

proof: Let $\mathcal{M} = \{E \in \mathcal{B} : \nu(E) \leq \mu(E)\}$. If $a < b$ then $(a, b] \in \mathcal{M}$ because

$$\nu((a,b]) = \int_a^b f = \int_a^b F_\mu' \leq F_\mu(b) - F_\mu(a) = \mu((a,b])$$

similarly $(-\infty, b]$ and $(a, +\infty) \in \mathcal{M}$. Since both μ, ν are finitely additive, each finite, disjoint union of the above sets are in $\mathcal{M}$. Call this collection $\mathcal{A}$. We have shown that $\mathcal{A}$ is an algebra. If we can show $\mathcal{M}$ is a monotone class then $\mathcal{M} \supset \mathcal{M}(\mathcal{A}) = S(\mathcal{A}) = \mathcal{B}$ and ~~then~~ Lemma would be proved.

Suppose $A_1 \supset A_2 \supset \dots$ are each in $\mathcal{M}$ since $\nu(A_1), \mu(A_1) < \infty$ we have $\nu(\cap A_n) = \lim \nu(A_n) \leq \lim \mu(A_n) = \mu(\cap A_n)$ and thus $\cap A_n \in \mathcal{M}$. Increasing chains are handled similarly. Therefore $\mathcal{M}$ is a monotone class.

Continuing, how we can form the Borel measure $\mu - \nu$. Since $F_{\mu-\nu} = F_\mu - F_\nu$, and $F_{\mu-\nu}' = F_\mu' - F_\nu' = F_\mu' - F_\mu' = 0$ a.e. and since ~~if~~ for $E \in \mathcal{B}$ $(\mu - \nu)(E) \leq \mu(E)$ we have:

Proposition 5. Each Borel measure μ can be written as the sum of two Borel measures $\lambda + \nu$ where $F_\mu' = F_\nu'$ a.e and $F_\lambda' = 0$ a.e.

Suppose we can prove the following

Proposition 6: If $\mu \ll m$ and $F_\mu' = 0$ a.e. then $\mu(E) = 0$ for each $E \in \mathcal{B}$

Theorem 7. The Borel meas μ has a density function, if and only if $\mu << m$

proof: ($\Rightarrow$) has already been done
($\Leftarrow$): If λ is as in the notation of Proposition 5, we have $F_\lambda' = 0$ a.e. Further more if $E \in \mathcal{B}$ with $m(E)=0$ then $\mu(E)=0$ by hypothesis; and since $\forall E \in \mathcal{B}$ $0 \le \lambda(E) \le \mu(E)$ we have $m(E)=0 \Rightarrow \lambda(E)=0$. Therefore, F_λ is absolutely continuous and by Prop 6 it must be constant. Therefore $F_\lambda \equiv 0$ & $\lambda \equiv 0$ so that $\mu \equiv \nu$, and ν has density F_μ'.

We will now start to prove Prop 6 & the fact that increasing functions are differentiable a.e.. Several preliminary thoughts are in order

DN 8: If $E \subset \mathbb{R}$ and $\mathcal{J}$ is a collection of non-trival intervals of $\mathbb{R}$ $\mathcal{J}$ is said to be a Vatali covering of E if $\forall x \in E$ and $\forall \varepsilon > 0$ $\exists I \in \mathcal{J}$ with $x \in I$ & $\ell(I) < \varepsilon$

Lemma 9 (Vatali): If $E \subset \mathbb{R}$ with $m(E) < \infty$ and $\mathcal{J}$ is a Vatali covering of E then for each $\varepsilon > 0$ there is a finite p.w. disjoint collection $I_1, \dots I_N$ of elements of $\mathcal{J}$ with $m(E \setminus (\bigcup_{i=1}^N I_i)) < \varepsilon$.

proof: We may assume each $I \in \mathcal{J}$ is closed (since the endpts have measure zero). Let U be an open set with $U \supset E$ and $m(U) < \infty$. Note that $\mathcal{J}' = \{I \in \mathcal{J} ; I \subset U\}$ is also a Vatali covering of E (since $e \in E$ implies that e is a positive distance from the closed set $\mathbb{R} \setminus U$.) Therefore we may assume that each $I \in \mathcal{J}$ is contained in U.

The idea of the proof is simple, we will inductively choose $\{I_n\}$ to be p.w. disjoint elements of $\mathcal{J}$, so that they take up as much "room" as "possible". Let $I_1 \in \mathcal{J}$ be arbitrary, and suppose $I_1, \dots I_n$ have been choosen so that

(1) $\{I_i\}_{i=1}^{n}$ are p.w.d elements of $\mathcal{J}$

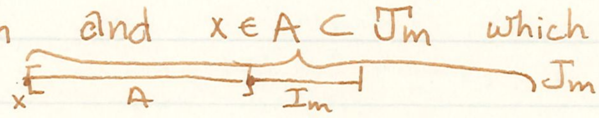
(2) $\ell(I_{j+1}) \geq \frac{1}{2} \sup\{\ell(I) : I \in \mathcal{J}, I \cap I_i = \emptyset \text{ for } i = 1, \dots j\}$.

(and let us define $k_j = \sup\{\ell(I) : I \in \mathcal{J}, I \cap I_i = \emptyset \text{ for } i = 1, \dots j\}$)

As long as $E \setminus (\bigcup_{i=1}^{n} I_i) \neq \emptyset$, we can choose such an I_{n+1}. (If $E \setminus (\bigcup_{i=1}^{n} I_i) = \emptyset$ the Lemma is proved.)

Since $\bigcup_{i=1}^{\infty} I_i \subset U \Rightarrow \sum_{i=1}^{\infty} \ell(I_i) \leq m(U) < \infty$ and thus $\sum_{i=j}^{\infty} \ell(I_i) \to 0$ and so does $k_j \to 0$, as $j \to \infty$. Pick N so that $\sum_{i=N+1}^{\infty} \ell(I_i) < \varepsilon/5$ and let $R = E \setminus (\bigcup_{i=1}^{N} I_i)$. we will complete the prove by showing $m(R) < \varepsilon$. We will do this by show $R \subset \bigcup_{i=N+1}^{\infty} J_i$, where J_i is the interval with the same midpoint as I_i and $\ell(J_i) = 5\ell(I_i)$. this will follow since then $m(R) \leq \sum_{i=N+1}^{\infty} \ell(J_i) = \sum_{i=N+1}^{\infty} 5\ell(I_i) < 5(\varepsilon/5) = \varepsilon$

Let $x \in R$ since $\bigcup_{i=1}^{N} I_i$ is closed, x must be a positive distance from the set $\bigcup_{i=1}^{N} I_i$, hence there is $A \in \mathcal{J}$, $\ell(A) > 0$ with $x \in A$ and $A \cap (\bigcup_{i=1}^{N} I_i) = \emptyset$. Since $k_j \to 0$ there must be some k_n with $\ell(A) > k_n$ and hence $A \cap I_i \neq \emptyset$ for some $i \leq n$ (by the definition of k_n). Let I_m be the first index with $A \cap I_m \neq \emptyset$. We have $m \leq n$, $m > N$ and $k_{m-1} \geq \ell(A)$ thus $\ell(A) \leq 2 I_m$ and $x \in A \subset J_m$ which completes the proof.



If $f : [a,b] \to \mathbb{R}$ define $V(f) = V_a^b(f)$ to be sup of the sums $\sum_{i=1}^{n} |f(a_i) - f(a_{i-1})|$ where $a = a_0 \le a_1 \le \dots \le a_n = b$. If $V_a^b(f) < \infty$ we will say f is of bounded variation and $V_a^b(f)$ is called the variation of f.

Note that if f is increasing (decreacing) $V_a^b(f) = f(b) - f(a)$ (resp. $f(a) - f(b)$) and it is easy to check that if f and g are increasing then $V_a^b(f \pm g) \le V_a^b(f) + V_a^b(g)$. Also if f is of bounded variation then $V(\alpha f) = |\alpha| V(f)$. The only thing keeping V from being a norm is that $V(f) = 0$ is equivalent to $f \equiv$ constant. Thus the space of functions f with bounded variation and $f(a) = 0$ is a norm space with norm $V(\cdot)$.

<u>Lemma:</u> f is of bounded variation on $[a,b]$ if and only if f is the difference of two increasing functions

pf ($\Leftarrow$) was done above

($\Rightarrow$) Suppose f is of bounded variation define two functions on $[a,b]$ $I(c)$ and $D(c)$ via

$I(c)$ (resp $D(c)$) is the sup of the sums like in $V_a^c(f)$ except $|f(a_i) - f(a_{i-1})|$ is replaced with $(f(a_i) - f(a_{i-1}))^+$ (resp $(f(a_i) - f(a_{i-1}))^-$). Clearly $I(c)$ and $D(c)$ are increasing functions.

I Claim $V_a^c(f) = I(c) + D(c)$ & $f(c) = I(c) - D(c) + f(a)$

Since (1) $(f(a_i) - f(a_{i-1})) = (f(a_i) - f(a_{i-1}))^+ - (f(a_i) - f(a_{i-1}))^-$

and (2) $|$ " $| =$ " $+$ "

the sums used in one of I, D, V are related in the same way

We use (1) by

$$f(c) - f(a) = \sum (f(a_i) - f(a_{i-1}))^+ - \sum (f(a_i) - f(a_{i-1}))^-$$

$$f(c) - f(a) + D(c) \geq \sum (f(a_i) - f(a_{i-1}))^+$$

$$f(c) - f(a) + D(c) \geq I(c)$$

so $$f(c) \geq I(c) - D(c) + f(a)$$

also $$f(a) - f(c) + \sum (f(a_i) - f(a_{i-1}))^+ = \sum (f(a_i) - f(a_{i-1}))^-$$

so $$f(a) - f(c) + I(c) \geq D(c)$$

similarly $V_a^c(f) = I(c) + D(c)$ (although $V_a^c(f) \geq I(c) + D(c)$ is a little trickier)

Cor 12: If f is of bdd variation, f' exist a.e.

Cor 13: If $f \in L_1(\mathbb{R})$ $F(x) = \int_{-\infty}^{x} f\,dx$, then F is of bounded variation

DN 14: Redefining abs cont of $F(x)$ by replacing $\sum F(x_i) - F(x_{i-1})$ by $\sum |F(x_i) - F(x_{i-1})|$

Lemma 15. F abs cont on $[a,b] \Rightarrow F$ is of bdd variation on $[a,b]$.

proof Let $\varepsilon = 1$, δ the resulting δ K an integer so that $K\delta \geq b-a$. Any sum used to find $V_a^b(f)$ can be refined by add the division pts $a+\delta, a+2\delta, \ldots$ and this increases the sum thus $V_a^b(f) \leq K$.

Note that both I & D defined Lemma 11 are abs cont if f is absolutely cont.

Prop. 10. If G is increasing and absolutely continuous on $[a,b]$ with $G'=0$ a.e on $[a,b]$ then G is constant on $[a,b]$.

Proof: Before ~~Proof~~ proving this we note that this will also prove Prop 6. Let $c \in [a,b]$, it suffices to show that for each $\omega > 0$ that $G(c) - G(a) < \omega$. So let $\omega > 0$ be given and let $\varepsilon_1 (b-a) < \omega/2$ and $\varepsilon_2 = \omega/2$.

Since $G' = 0$ a.e. on $[a,c]$, let E be the set with $m(E) = c-a$, $G' = 0$ on E & $a, c \notin E$. For each $\delta > 0$ and each $e \in E$ there is $e' \in (a,c)$ with $|e-e'| < \delta$ and $G(e) - G(e') < \varepsilon_1 (e-e')$ since $G'(e) = 0$. Thus the collection of such intervals $[e,e']$ (or $[e',e]$) form a Vitali Covering of E. Let $\delta > 0$ be the number resulting from G being absolutely continuous with $\varepsilon = \varepsilon_2$.

By Lemma 9, there are $a_1, \ldots a_N, b_1, \ldots b_N$ with $a \le a_1 < b_1 < a_2 < b_2 \ldots < a_N < b_N \le c$ with

(1) $G(b_i) - G(a_i) \le \varepsilon_1 (b_i - a_i)$ and

(2) $m\left(E \setminus \left(\bigcup_{i=1}^{n} [a_i, b_i]\right)\right) < \delta$

Hence (3) $(a_1 - a) + \sum_{i=1}^{n-1} (a_{i+1} - b_i) + (c - b_n) < \delta$

and (4) $G(c) - G(b_n) + \sum_{i=1}^{n-1} [G(a_{i+1}) - G(b_i)] + G(a_1) - G(a) < \varepsilon_2$ (by abs cont)

By (1): (5) $\sum_{i=1}^{n} (G(b_i) - G(a_i)) \le \varepsilon_1 \sum_{i=1}^{n} (b_i - a_i) \le \varepsilon_1 (c-a) < \omega/2$

thus by summing (4) & (5) $G(c) - G(a) < \varepsilon_2 + \omega/2 < \omega$, which completes the proof.

Our final result we have been postoning for weeks is

Theorem ??: If f is increasing $: \mathbb{R} \to \mathbb{R}$ then f' exists a.e.

proof: To say take $f'(x)$ exists says that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists. since limits can fail to exist we can rewrite this condition in terms of things that do always exist as follows define

$$D^+f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \qquad D^-f(x) = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

$$D_+f(x) = \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \qquad D_-f(x) = \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$$

clearly $D^+f(x) \geq D^-f(x)$ and they are equal if and only if f has a right-hand derivative at x. Similarly $D_+f(x) \geq D_-f(x)$ with $=$ equivalent to f having a left-hand derivative at x. And $f'(x)$ exists exactly if the 4 numbers $D^\pm f(x)$, $D_\pm f(x)$ are are equal. [we do not worry about $\pm\infty$ for a value of $f'(x)$ since we have shown (assuming that the limit exist a.e. as an extended real) $f'(x) = \pm\infty$ at most on a set of measure zero.]

Thus the set $B = \{x : \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ fails to exist as an extended real$\}$ can be written as the union of three sets $B = \{x : D^+f(x) > D^-f(x)\} \cup \{x : D_+f(x) > D_-f(x)\} \cup \{x : D^+f(x) \neq D_-f(x)\}$. It suffices to show that each of these three sets as measure zero. We will show that the first set has measure zero, the others are similar (~~the~~ Compare with Rogden which shows that the last set has measure zero.)

We first reduce the problem even further, let

$W = \{(u,v) : u, v \in \mathbb{Q},\ u > v\}$. As a subset of $\mathbb{Q}\times\mathbb{Q}$, W is countable. We claim that

$$\{x : D^+f(x) > D^-f(x)\} = \bigcup_{(u,v)\in W} \{x : D^+f(x) > u > v > D^-f(x)\}$$

It is easy to see that the right hand side $\subset$ left hand side. Conversely, the density of the rationals implies that if $D^+f(x) > D^-f(x)$ then there are rationals u, v with $D^+f(x) > u > v > D^-f(x)$. Since W is countable it suffices to show for each $(u,v) \in W$, $E = \{x : D^+f(x) > u > v > D^-f(x)\}$ has measure zero.

Suppose $m(E) = \lambda > 0$ and let $\varepsilon > 0$, let U open $\supset E$ with $m(U) < \lambda+\varepsilon$ (assume $2\varepsilon < \lambda$ without loss of generality). Since $x \in E$ implies $D^-f(x) < v$, for each $\delta > 0$ $\exists h$, $0 < h \le \delta$ with $f(x+h) - f(x) < vh$ and $[x, x+h] \subset U$. Thus the set of such $[x, x+h]$ form a Vitali covering for E. By the Lemma $\exists x_1, \dots x_N, h_1, \dots, h_N$ with $\{[x_i, x_i+h_i]\}_{i=1}^N$ and $m\left(E \setminus \left(\bigcup_{i=1}^N [x_i, x_i+h_i]\right)\right) < \varepsilon$. Let $F = E \cap \bigcup_{i=1}^N [x_i, x_i+h_i)$. Clearly $m(F) > \lambda - \varepsilon$ and $\sum_{i=1}^N h_i > \lambda - \varepsilon$ and we have $\sum_{i=1}^N (f(x_i+h_i) - f(x_i)) \le v\sum_{i=1}^N h_i < v(\lambda+\varepsilon)$ since $\bigcup_{i=1}^N [x_i, x_i+h_i] \subset U$.

For each $y \in F$, $\delta > 0$ $\exists k$, $0 < k < \delta$ with $[y, y+k] \subset \bigcup_{i=1}^N [x_i, x_i+h_i]$ so that $f(y+k) - f(y) > uk$. ~~where~~ Thus the set of all such $[y, y+k]$ form a Vitali covering of F. So there are $y_1, \dots y_M$, $k_1, \dots k_M$ with $\{[y_j, y_j+k_j]\}_{j=1}^M$ p.w.d. $\bigcup_{j=1}^M [y_j, y_j+k_j] \subset \bigcup_{i=1}^N [x_i, x_i+h_i]$ and $m\left(F \setminus \left(\bigcup_{j=1}^M [y_j, y_j+k_j]\right)\right) < \varepsilon$. Thus $\lambda - 2\varepsilon < m\left(\bigcup_{j=1}^M [y_j, y_j+k_j]\right) = \sum_{j=1}^M k_j$. We have

$(\lambda - 2\varepsilon)u < u\sum_{j=1}^M k_j < \sum_{j=1}^M (f(y_j+k_j) - f(y_j)) \le \sum_{i=1}^N (f(x_i+h_i) - f(x_i)) \le v\sum_{i=1}^N h_i < v(\lambda-\varepsilon)$ or $(\lambda-2\varepsilon)u < v(\lambda-\varepsilon)$ since this is true $\forall \varepsilon > 0$ $u \le v$ a contradiction.