Honors Day 2016
$V - E + F = 2$

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In a letter to Goldbach, Euler names edges:

"the junctures where two faces come together along their sides, which for lack of an accepted term, I call edges."

The E in the formula $V - E - F = 2$.

Not the first e, Euler also is responible for e, the base of the natural log and the beautiful formula

$$e^{i\pi} = -1$$

Polyhedra Definitions

- Polygon has sides and plane angles and corners.
- An edge is shared by two polygon sides and has a dihedra angle.
- Vertices have a solid angle and is shared by three or more corners.
- Either a solid or a manifold condition.

Timeline

c 570BC	Pythagoras – Dodecahedron
417-369BC	Theatetus – the five regular solids Icosahedron, Octohedron.
428-347BC	Plato The five Platonic solids.
c 300BC	Euclid Elements – final book.
287-212BC	Archimedes: Archimedean solids 13 polyhedra in lost work
1596-1650	Descartes
1751	Euler
1794	Legendre
1811	Cauchy
1813	Linuiler – examples that fail
1847	Listing introducted the word Topology
1854-1912	Poincare Analysis Situs
1875-1941	Lebesgue
1905	Sommerville discovers # 14
1912	Hausdroff First Topology Textbook

Euler's Polyhedra Formula

A convex polyhedron with V vertices, E edges and F faces satisfies $V - E + F = 2$.

A connected planar graph with V vertices, E edges and F faces satisfies $V - E + F = 2$. (The unbounded region counts as a face.)

The five regular solids

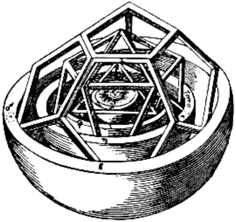
$(3,3)$ Tetrahedron $V = 4, E = 6, F = 4$

$(4,3)$ Cube $V = 8, E = 12, F = 6$

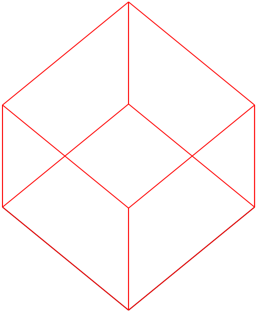
$(5,3)$ Dodecahedron $V = 20, E = 30, F = 12$

$(3,4)$ Octahedron $V = 6, E = 12, F = 8$

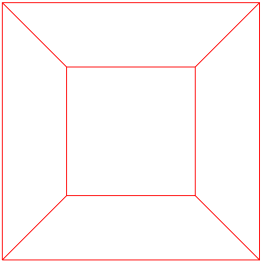
$(3,5)$ Icosahedron $V = 12, E = 30, F = 20$



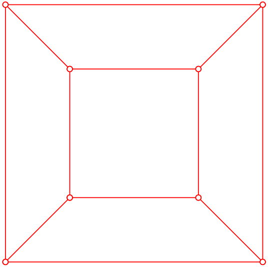
Cube: $V = 8, E = 12, F = 6$



Cube 3D



Flattened cube



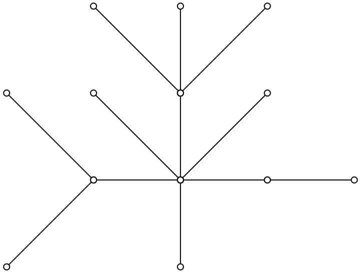
Graph Theory Cube

Tree

A graph is a tree $\iff$ it is connected and acyclic.

Every connected graph G contains a spanning tree T, a subgraph which is a tree and $V_G = V_T$ (the definition of T spans G).

Tree $V = n, E = n - 1, F = 1$



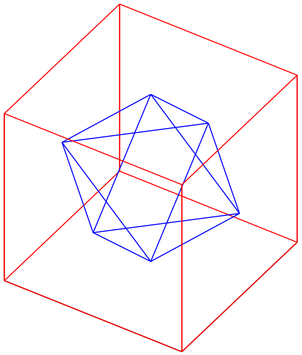
Induction: remove a leaf, decreases both V and E by one.

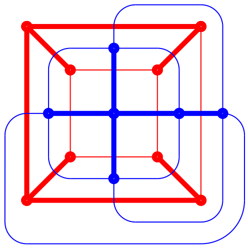
Duality

Polyhedra, pick vertices in the middle of faces, convex hull.

Graph theory. pick vertices in faces, connect adjacent faces. The resulting faces contain the orginal vertices.

Duality



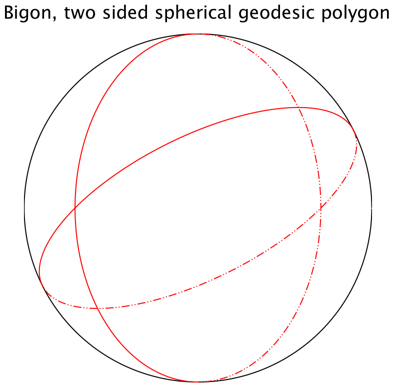


Let T be a spanning tree and S the dual tree of the planar graph G

$$V_T - E_T = 1$$
$$V_S - E_S = 1$$
$$V_G - E_G + F_G = 2$$

Since $E_G = E_T + E_S$, $F_G = V_S$ and $F_G = V_S$

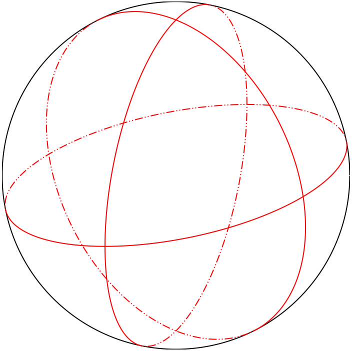
Area of bi-gon, $n = 2$



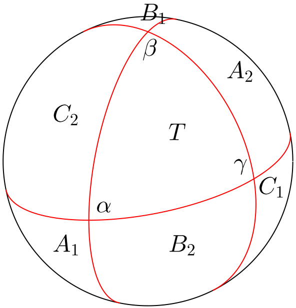
The area is the fraction of the hemi-sphere $= \frac{\alpha}{\pi} 2\pi$ or

$$\text{Area } = 2\alpha = \alpha + \alpha - (n-2)\pi$$

Geodesic Spherical Triangle



Area of triangle, $n = 3$



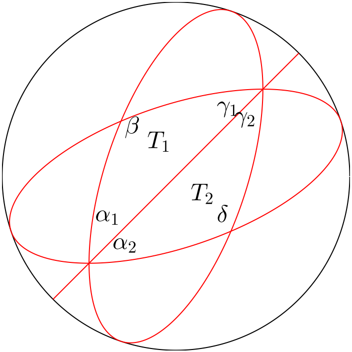
Area of triangle, $n = 3$

$$
\begin{aligned}
T + A_1 + A_2 &= 2\alpha \\
B_1 + B_2 &= 2\beta - T \\
C_1 + C_2 &= 2\gamma - T \\
2\pi &= 2(\alpha + \beta + \gamma - T)
\end{aligned}
$$

$$T = \alpha + \beta + \gamma - (n - 2)\pi$$

Area of n-gon

$$\text{Area } = T_1 + T_2 = \alpha + \beta + \gamma + \delta - (n-2)\pi$$



$$T_1 = \alpha_1 + \beta + \gamma_1 - \pi$$
$$T_2 = \gamma_2 + \delta + \alpha_2 - \pi$$
$$\text{Area} = \alpha + \beta + \gamma + \delta - 2\pi$$

Legendre's proof

$$
\begin{aligned}
\text{Area of Sphere} &= \sum_F \text{Area of F} \\
4\pi &= \sum_F \Big(\sum_{\text{angles}} \alpha - (n_F - 2)\pi \Big) \\
4\pi &= \sum_V \Big(\sum_{\text{angles}} \alpha \Big) - \pi \sum_F n_F + 2|F|\pi \\
4\pi &= |V|2\pi - \pi 2|E| + 2|F|\pi \\
2 &= |V| - |E| + |F|
\end{aligned}
$$

Results, a quote and a joke

- The average degree of a planar graph is strictly less than six.
- Soccer ball like polyhedra have exactly 12 pentagons.
- Mathematics is the art of giving the same name to different things – Poincare.
- Without Geometry, life is pointless.