Honors Day 2017
Euclid takes the 5th

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Honors Day
Florida State University, Tallahassee, FL
April 7, 2017

Euclid

Little is known about Euclid's life.

Most famous for the Elements (c 300BC), and it is best known for its geometric results. The 5th postulate is the parallel postulate.

Book 1 contains 48 Propositions, the first 28 don't use the 5th and the first 15 are true in spherical geometry too. Euclid delayed using the 5th as long as possible.

The geometry textbook until at least 1900.

The Common notions

The common notions are: if $a = b, b = c, d = e$, then

1. $a = c$
2. $a + d = b + e$
3. $a - d = b - e$
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

Postulates

1. To draw a straight line from any point to any other.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to each other.
5. That if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, if produced indefinitely, meet on the side on which the angles are less than the two right angles.

Time line

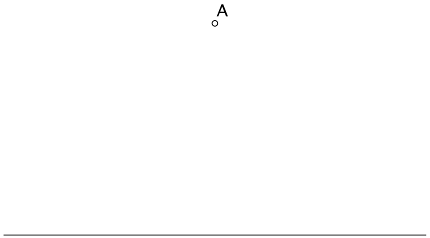
c 300BC	Euclid Elements
410-485	Proclus
1663	Wallis
1697	Saccheri
1794	Legendre
1795	Playfair
1817	Gauss
1823	Bolyai
1829	Lobachevsky
1868	Bertrami
1871	Klein
1881	Poincaré
1899	Hilbert
1930	Tarski

Take another 5th

Playfair	Given a line and a point not on the line, it is possible to draw exactly one line through the given point parallel to the line.
Wallis	To each triangle, there exists a similar triangle of arbitrary magnitude
Legendre	The sum of the angles of a triangle is equal to two right angles.

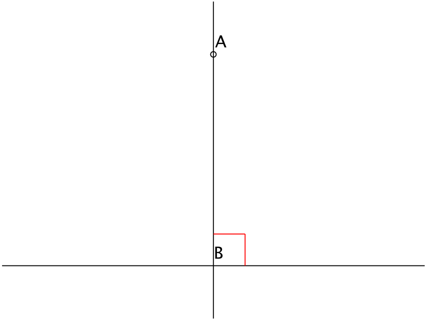
Proclus (410-485) knew Playfair (1748-1819) axiom was equivalent to the 5th. Wallis (1616-1703) was a false proof. Legendre (1752-1833) spend 40 years working on the 5th.

Existence of Parallel Lines



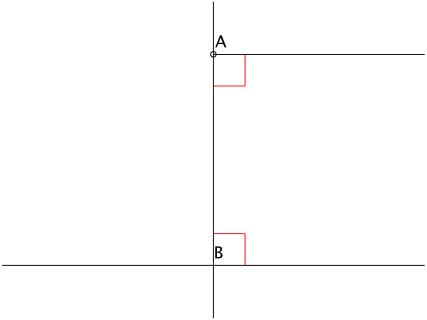
Given a line ℓ and a point A not on the line construct a line through A parallel to ℓ.

Prop 12: Perpendicular bisector



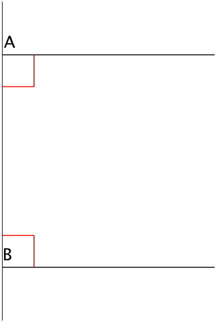
Given a line ℓ and a point A not on the line construct a line through A perpendicular to ℓ.

Prop 11: Perpendicular bisector



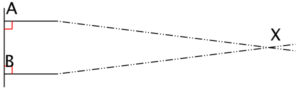
Given a line ℓ and a point A on the line construct a line through A perpendicular to ℓ.
Prop 17 implies these lines are parallel: The sum of any two angles of a triangle is less than two right angles.

Parallel Lines



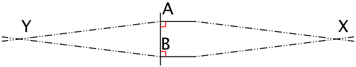
Suppose not, then the lines will mean at a point X making an isosceles triangle.

Prop 6: isosceles triangles have equal sides



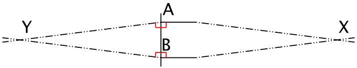
Suppose not, then the lines will mean at a point X making an isosceles triangle.

Postulate 3



Use the length $AX = BX$ to find Y, Prop 8 (SSS) says angles YAB and YBA are right angles

Prop 14: both YAX and YBX are lines

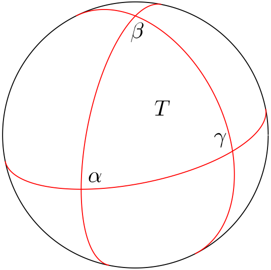


Postulate 1 means there is a unique line between Y and X.

Spherical Geometry

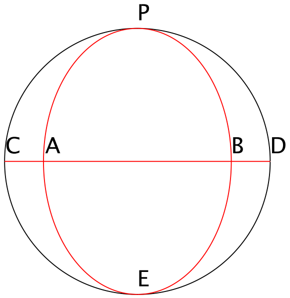
Line is a geodesic, a great cirlce.
Points are pairs of points on opposite sides of the sphere.
These are called antipodal points.

Spherical Geometry



$$\text{Area of T } = \alpha + \beta + \gamma - \pi$$

Similar triangles are congruent.



If $\angle APB = \angle PAB = \angle PBA = \pi/2$, then the area of PAB is $1/8$ of the surface area of the sphere namely $\pi/2$.
If $\angle CPD = \angle PCE = \angle PDE = \pi$, then the area of everything is 2π. The biggest triangle.

Bolyai, Lobachevsky and Gauss

Bolyai (1823 idea, 1832 published)

Lobachevsky (1826 idea, 1829 published)

Gauss (1817 idea, never published) coined Non-Euclidean Geometry.

A revolution in geometry. Lobachevsky is called the Copernicus of geometry in Russia. Gauss wanted to avoid causing a fuss, it was a revolutionary denial of Euclid. Kant had made Euclid almost god like. Bolyai got depressed and suspicious.

Beltrami, Klein, Poincaré

Beltrami The pseudo-sphere is a model of (part) of the hyperbolic plane.

Klein The names elliptical, hyperbolic, parabolic. The chord model (which is due to Beltrami).

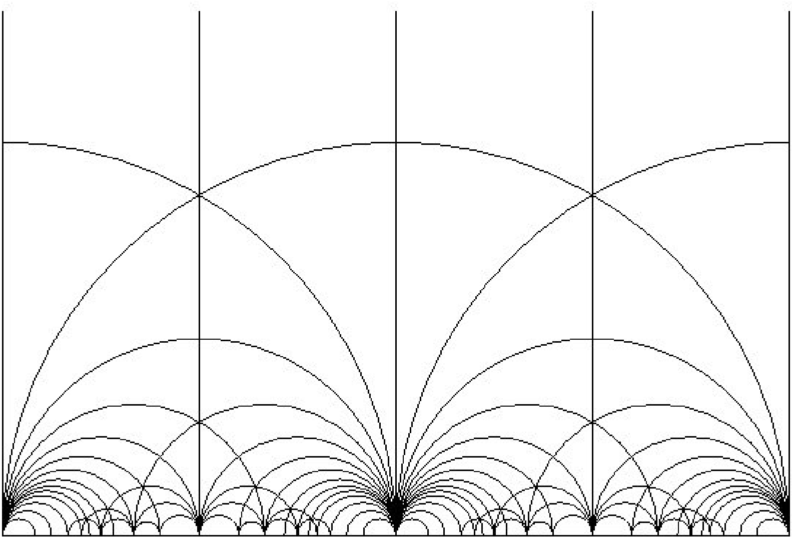
Poincaré The disk model and the upper half plane model (both also due to Beltrami).

Klein's Erlangen Program. Poincaré results were better known than Beltrami earlier papers. The revolution is complete.

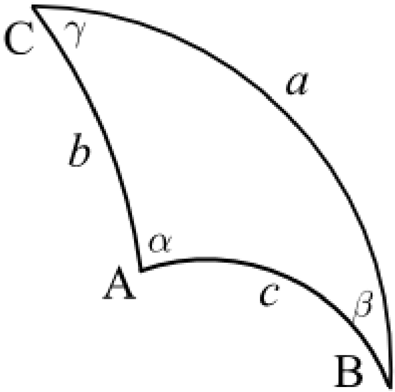
Escher's Circle Limit III

From Wikipedia

Half plane model of hyperbolic geometry



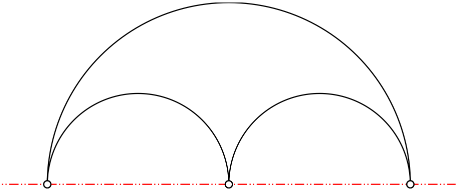
Hyperbolic Triangles



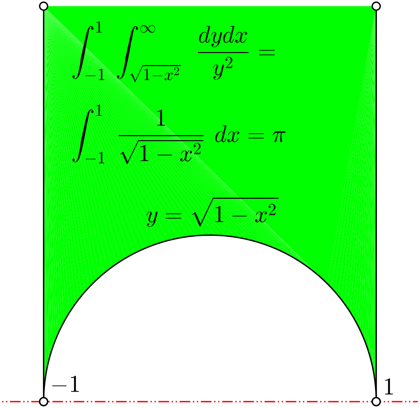
$$\text{Triangle Area} = \pi - \alpha - \beta - \gamma$$

Similar triangles are congruent

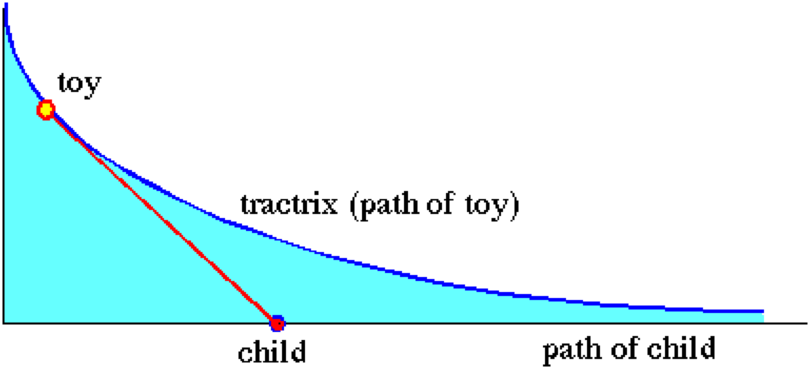
The largest triangle



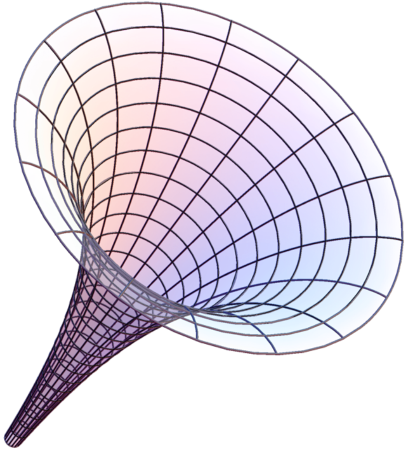
All “angles” are 0, area is π.


$\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}\frac{dydx}{y^2} =$
$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\,dx = \pi$
$y = \sqrt{1-x^2}$
-1
1

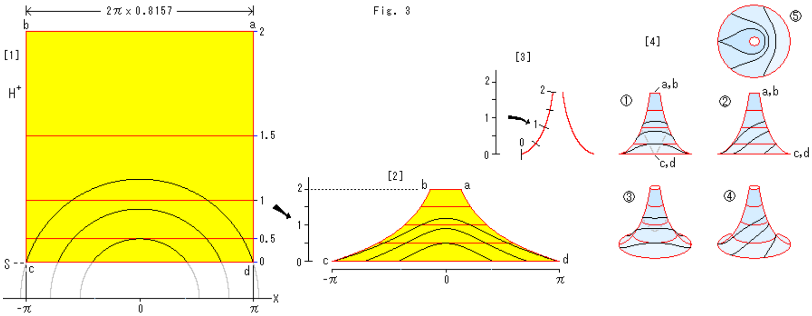
Tractrix



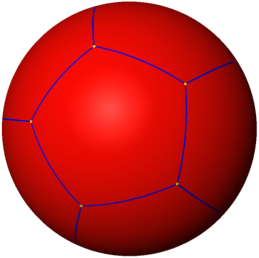
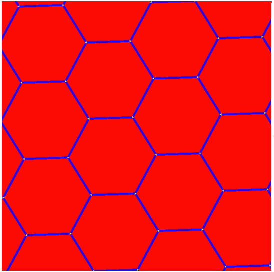
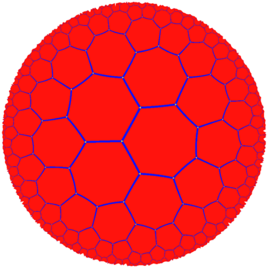
Pseudo-sphere



Pseudo-sphere and Half-plane



Kale

Drawings from Wikipedia by Tomruen

Hilbert’s axioms for geometry (1899)

1. Incidence (example: there are at least two points on a line, there exist at least three points that do not lie on a line.)
2. Order (example: Of any three points on a line there exists no more than one that lies between the other two.)
3. Congruence
4. Parallels
5. Continuity (Infinities and Infinitesimals are impossible)

Tarski's axioms for geometry (1930, 1999)

A first-order complete consistent and decidable version of geometry. There are only points, no lines in the axioms.

- Betweenness a triadic relation *Bprq*, the point r is on the line segment $[pq]$.
- Congruence $pq = rs$, they have the same length.
- p, q, r are co-linear if (exactly) one of *Bpqr*, *Bqrp*, or *Brpq* is true.
- One Axiom schema of Continuity essentially Dedekind cuts.
- Three points equal distance from two distinct points lie on a line.

Parallel axiom Versions

A A line segment joining the midpoints of two sides of a triangle is half the length of the third size.

B Given any triangle there is a circle that includes its vertices.

C Given any angle and any point v in its interior, there is a line segment including with endpoints on each side of the angle.