(1) Consider an undirected network with \( n \) vertices and adjacency matrix \( A \). Suppose the edges are weighted, each with weight \( \alpha \).

- What is the total number of paths of length \( r \) between two vertices \( s \) and \( t \)?

- What is the sum of the weights of paths of length \( r \) between \( s \) and \( t \)?

- What is the sum of the weights of all paths of length \( r \) or less between \( s \) and \( t \)?

- What does this last sum converge to in the limit \( r \to \infty \)? (Hint: you did this in calculus, but with a scalar variable \( x \) rather than a matrix)
(2) The vertex degrees can be collected into a vector $\vec{k}$.

- Suppose that the degree vector for a simple undirected (and unweighted) network is $\vec{k} = (2, 2, 3, 3, 2)$. Give a plausible adjacency matrix $A$ for the network.

- Next consider a simple directed network with in-degree and out-degree vectors $\vec{k}^{\text{in}} = (3, 3, 2, 3, 2)$ and $\vec{k}^{\text{out}} = (3, 3, 3, 3, 1)$. Give a plausible adjacency matrix $B$.

- What is the network density for each of these networks?
What is the size $k$ of the minimum vertex cut set between $X$ and $Y$ in the network below? Prove your result by finding one possible cut set of size $k$ and one possible set of $k$ vertex-independent paths between $X$ and $Y$. Why do these two actions constitute a proof that the minimum cut set has size $k$?
(4) Consider diffusion on a connected network with \( n \) vertices and diffusion coefficient \( C \). Then the spectral solution of the diffusion equation on the network is

\[
\vec{\psi}(t) = \sum_{j=1}^{n} a_j(0)e^{-C\lambda_j t} \vec{v}_j.
\]

- What do we know about the eigenvalues \( \lambda_j \) and eigenvectors \( \vec{v}_j \) of the graph Laplacian?

- How can you determine values of \( a_j(0) \)?

- What happens to \( \vec{\psi}(t) \) in the limit \( t \to \infty \)?

- Suppose that \( \vec{\psi}(0) = \kappa \vec{v}_3 \). What would be the values of the \( a_j(0) \)? Write down the spectral solution in this case.

- In this case, what would \( \vec{\psi}(t) \) approach as \( t \to \infty \) and at what rate?