Assignment 4 (Graph Theory and Networks)  
Due on November 2

(1) (a) Determine the modularity of a bisection of the complete graph $K_5$ in 
which one cluster contains two nodes and the other contains three nodes. 
(b) Generalize this for all possible bisections of $K_5$. That is, give a formula 
for modularity for a bisection in which one cluster contains $N_1$ nodes and 
the other contains the remaining nodes. 
(c) Generalize this further to the complete graph $K_N$. What is the modularity 
of this graph when one cluster of the bisection contains $N_1$ nodes and the 
other cluster contains the remaining nodes? Show that this is always 
negative, regardless of the bisection used. [Since the partition of a single 
cluster has modularity zero, it is also the maximum modularity partition. 
In other words, modularity optimization does not split cliques.]

(2) Suppose that $A$ and $B$ are two of the $q > 2$ clusters of a network partition, with 
degrees $k_A$ and $k_B$, respectively. Also suppose that these are approximately 
the same, so $k_A \approx k_B = k$. Let $L_A$, $L_B$, and $L_{AB}$ be the number of links inside 
cluster $A$, inside cluster $B$, and between $A$ and $B$, respectively. Compute the 
difference in modularity between this partition and the one in which $A$ and 
$B$ are merged, called cluster $C$. What condition on $k$ makes the partition 
with $A$ and $B$ merged have larger modularity than the one in which they are 
separated?

(3) A bipartite affiliation graph that shows the membership of people in different 
social foci can be projected into a graph that just shows the people, in which 
two people who share a common focus are joined by an edge. An example 
of such a projection is shown below. For this projection, draw a bipartite 
affiliation network consistent with this projection and having the minimum 
number of foci possible. Explain how you know that this is the minimum 
number.
(4) In the social network below, calculate the neighborhood overlap of all edges and place tie strengths on the edges in such a way that greater tie strengths are associated with greater neighborhood overlap. Are there any local bridges? If so, list them.

(5) The complementary cumulative distribution function $P_k$ of a discrete probability distribution $p_k$ is defined as $P_k = \sum_k^\infty p_k$. Show that the complementary cumulative distribution function of a power-law distribution $p_k = Ck^{-\alpha}$ with exponent $\alpha > 1$ and $k = 1, 2, ...$ is itself a power-law distribution. What is the exponent of the $P_k$ distribution? (Hint: approximate summation with integration.)