Assignment 4 (MAM1)
Due on October 23

(1) Constructing phase portraits.
For the following system find steady states, use linear stability analysis to classify them, sketch nullclines, and draw a phase portrait. Also, indicate the basins of attraction of any stable steady states and, for saddle points, indicate the stable and unstable manifolds near the saddle point.

\[
\begin{align*}
\dot{x} &= x(3 - x - y) \\
\dot{y} &= y(2 - x - y)
\end{align*}
\]

(2) Another phase portrait.
Consider the system \( \dot{x} = y^3 - 4x, \dot{y} = y^3 - y - 3x \).
(a) Find and classify all equilibria.
(b) Show that the line \( y = x \) is invariant, i.e., any trajectory that starts on it stays on it.
(c) Show that \( |x(t) - y(t)| \to 0 \) as \( t \to \infty \) for all other trajectories.
(d) Sketch a phase portrait.

(3) Determining the order of an expression.
Determine the order (big O) of the following expressions as \( \epsilon \to 0 \):
(a) \( \ln(1 + 5\epsilon) \)
(b) \( \frac{1 - \cos \epsilon}{1 + \cos \epsilon} \)
(c) \( 1 - \frac{1}{2} \epsilon^2 - \cos \epsilon \)
(d) \( \sqrt{\epsilon(1 - \epsilon)} \)
(e) \( \frac{\epsilon \sqrt{\epsilon}}{1 - \cos \epsilon} \)

(4) Lindstedt-Poincaré approach.
This problem uses the Lindstedt-Poincaré method on a nonlinear ordinary differential equation. At some point you’ll want to use the following trig identity: \( \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \). Consider the equation

\[
\ddot{x} + \omega_0^2 x = \epsilon \dot{x}^2 x 
\]

where \( \omega_0 \) is the frequency of the reduced-system oscillator.
(a) Determine a first-order uniform asymptotic approximation to the solution using the Lindstedt-Poincaré technique.
(b) Find an approximation that satisfies \( x(0) = 0, \dot{x}(0) = \beta \).