We will use a planar model for neuronal activity developed in the 1980’s by Jim Hindmarsh and Malcolm Rose, at Cardiff University. They were studying snail neurons, and wanted a model that could produce electrical impulses with lower dimension than the Hodgkin-Huxley model and written in terms of polynomials. This is now called the *Hindmarsh-Rose model* and was published in *Nature*, vol. 296, pp. 162–164, 1982. This planar model is not biophysical in the sense that it does not include ionic currents explicitly. However, it captures the excitable dynamics exhibited by the neuron. The model is:

\[
\begin{align*}
\frac{dv}{dt} &= (w - v^3 + 3v^2 + I_{ap})/c \\
\frac{dw}{dt} &= 1 - 5v^2 - w,
\end{align*}
\]

where \(v\) is voltage and \(w\) is a recovery variable (it can be positive or negative). The two parameters are applied current \(I_{ap}\) and capacitance \(c\).

**Exercises**

1. Write an XPPAUT program for the Hindmarsh-Rose model. Consider first the case with \(c = 2\). For \(I_{ap} = -2\), construct nullclines and classify the equilibrium (or equilibria) as stable/unstable node/spiral, etc. Record eigenvalues. Sketch a phase portrait, including nullclines, equilibria, and a few trajectories and turn it in.

2. Now set \(I_{ap} = -0.8\). Sketch the nullclines. Locate and classify the equilibria and limit cycles. For saddle points, you will be asked if you want to draw invariant sets. These are the stable and unstable manifolds. Say yes. Include these invariant sets in your sketch (there will be several curves sketched, so using color helps a lot). Be sure to identify which branches make up the stable and which make up the unstable manifold. Describe the basins of attraction of the stable equilibria. On a separate graph, plot \(v\) versus \(t\) for several initial conditions near each equilibrium. What kind of bifurcation occurred between \(I_{ap} = -2\) and \(I_{ap} = -0.8\)? Explain.

3. Now set \(I_{ap} = 0\). Sketch the nullclines. Locate and classify equilibria and limit cycles. Draw and label stable and unstable manifolds of the saddle point. Identify the basins of attraction for each stable structure. What do the branches of the unstable manifold converge to? Plot \(v\) versus \(t\) for several
initial conditions near each equilibrium. What kind of bifurcation occurred between $I_{ap} = -0.8$ and $I_{ap} = 0$? Explain.

(4) Set $I_{ap} = 2$. Sketch the nullclines. Locate and classify equilibria and limit cycles. What kind of bifurcation occurred between $I_{ap} = 0$ and $I_{ap} = 2$? Explain.

(5) Set $I_{ap} = 8$. Sketch the nullclines. Locate and classify equilibria and limit cycles. Plot $v$ versus $t$ for an initial condition near the equilibrium. Why does it take so long for transient behavior to die out? What kind of bifurcation occurred between $I_{ap} = 2$ and $I_{ap} = 8$? Explain.

(6) Use the Auto feature in XPPAUT to construct a bifurcation diagram with $I_{ap}$ as the bifurcation parameter. Start from $I_{ap} = -2$ and continue out to $I_{ap} = 10$. Turn in a computer plot of the diagram. Label all bifurcation points (SN=saddle node, TR=transcritical, subPF=subcritical pitchfork, supPF=supercritical pitchfork, subHB=subcritical Hopf, supHB=supercritical Hopf, HC=homoclinic). For what values of the bifurcation parameter is the system bistable?

(7) Construct and turn in a bifurcation diagram showing period vs. $I_{ap}$. To do this, in the AUTO window click Axes and then Period. Enter $Y_{min}=0$ and take a guess at what $Y_{max}$ might be. Click OK and then, back in the main AUTO menu, click ReDraw. If your guess for $Y_{max}$ was on target you should see the plot of period versus $I_{ap}$. If your $Y_{max}$ is too small you’ll need to set a larger $Y_{max}$. At what value of $I_{ap}$ is the period at a minimum? At a maximum? Is this a type 1 or type 2 oscillator?

(8) Redo parts 6 and 7, but with the capacitance reduced to $c = 1$. Label all bifurcations. (You will need to go past $I_{ap} = 10$ to find the second Hopf bifurcation.) What types of bifurcations are there now? Is this a type 1 or type 2 oscillator?

(9) Make yet another bifurcation diagram, but now with $c = 3$. What happened to the Hopf bifurcations and the periodic branch when $c$ was changed from 2 to 3?

(10) You have now created three different 1-parameter bifurcation diagrams (with $I_{ap}$ as bifurcation parameter), using three different values of the second parameter $c$. To do a more complete analysis you can construct a two-parameter bifurcation diagram. This is a curve in the plane of two parameters consisting of bifurcation points. These diagrams summarize the family of one-parameter bifurcation diagrams as a second parameter is varied. For example, one could trace out a curve of Hopf bifurcations as a second parameter (like $c$) is varied. Try this out. Reconstruct the 1-parameter bifurcation diagram of part 6 so that $V$ and $I_{ap}$ are on the axes. Grab one of the Hopf bifurcations. Click on Axes, and then Two par. You should see that $I_{ap}$ is
the main parameter and $c$ is the second parameter (if not, then make it so). You can also enter the max and min values for the two-parameter diagram. (Enter values that you think are appropriate, you can change them after the diagram has been created. You want your diagram to contain all the important information.) Next, click on Run and Two Param. This should generate the diagram. (You may need to run twice, with $D$s positive and then with $D$s negative, to get both parts of the curve.) This curve shows the set of $I_{ap}$ and $c$ parameters where a Hopf bifurcation occurs. (You should get the same curve if you start from the other Hopf bifurcation point in the one-parameter bifurcation diagram.)

In addition to the two-parameter continuation of the Hopf bifurcation, do a two-parameter continuation of the two SN bifurcations. Put all of these curves on the same figure and turn it in. Make sure to label all curves. To answer the following questions, consider only $c > 1$. For what region of the parameter plane is there at least one stable equilibrium? (You could answer this by coloring the appropriate portion of the plane.) Where is there bistability between two stable equilibria? Where is there stable periodic motion? Where is there bistability between a steady state and a periodic solution?