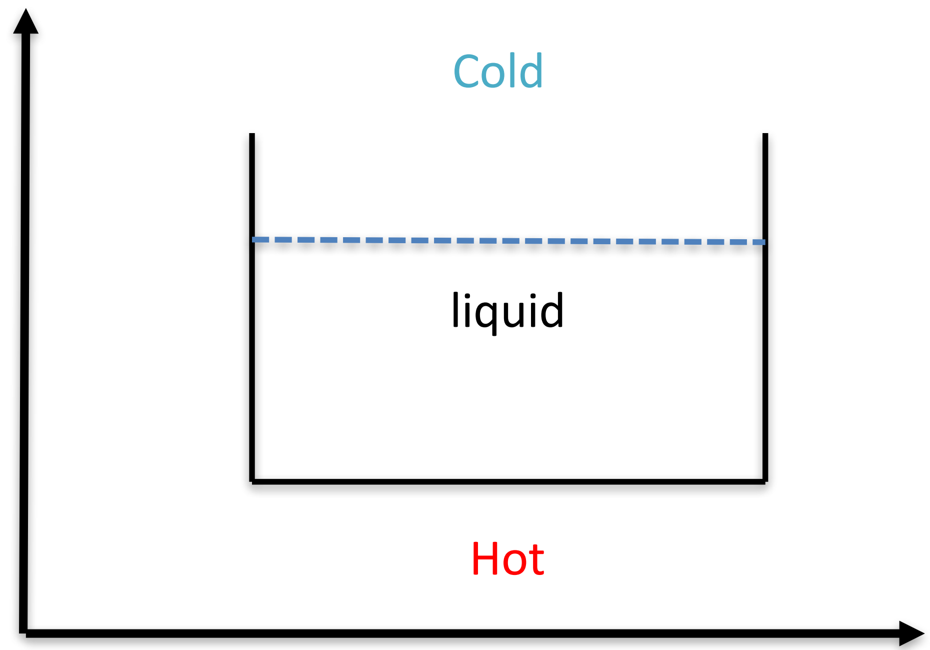
Chaos

The Lorenz Equations

Two-Dimensional Fluid Flow



Analog of the earth's atmosphere

The Lorenz Equations

The solution can be understood using Fourier analysis, with an infinite number of coefficients to be determined. That is, a system with an infinite number of non-linear ordinary differential equations. In 1963, Ed Lorenz simplified this to 3, by setting all others to constants.

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

x = "convective overturning"

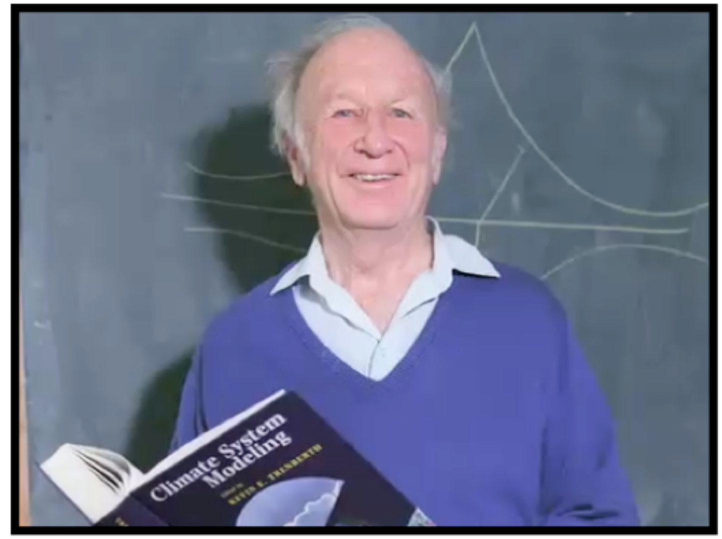
y = horizontal temperature variation

z = vertical temperature variation

σ = Prantl number

r = Rayleigh number

b = related to physical size of the system



Ed Lorenz
MIT

Heat Conduction Pathways

The heat introduced from below is transported up in two ways: thermal diffusion (stationary fluid) and thermal convection (fluid in motion).

Rayleigh number (r) = $\dfrac{\text{time scale for thermal diffusion}}{\text{time scale for thermal convection}}$

If r is low, then the heat is quickly dissipated through diffusion and the fluid remains stationary. Past some critical value of r, the fluid begins to move and forms convection rolls. For r significantly larger the fluid becomes turbulent (e.g., boiling water).

Lorenz examined the behavior of his simple model as r was increased from low to high.

Equilibria of the Lorenz Equations

In the case in which heat dissipation dominates, there is no convective overturning ($x = 0$), and no vertical or horizontal temperature variation ($y = z = 0$). This homogeneous solution is the only equilibrium when $r < 1$. He used $\sigma = 10$ and $b = {}^{8}/_{3}$.

There is a supercritical pitchfork bifurcation at $r = 1$, beyond which the two stable non-homogeneous equilibria are:

$$\vec{C}^{+} = \begin{bmatrix} \sqrt{b(r-1)} \\ \sqrt{b(r-1)} \\ r-1 \end{bmatrix} \quad \text{and} \quad \vec{C}^{-} = \begin{bmatrix} -\sqrt{b(r-1)} \\ -\sqrt{b(r-1)} \\ r-1 \end{bmatrix}$$

These reflect fluid motion and heat is transmitted through convection as well as diffusion.

These equilibria both lose stability at a Hopf bifurcation at $r_H \approx 24.74$.

What is the behavior for $r > r_H$?

Trajectories Remain Bounded

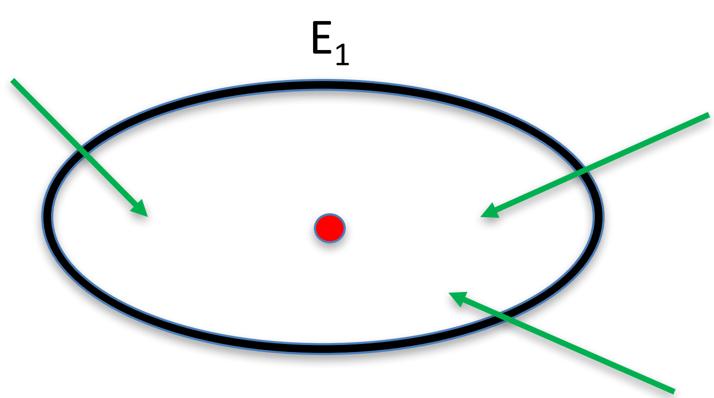
When $r > r_H$ it is possible that solutions tend to $\pm\infty$. We can use a Lyapunov-like function to show that this does not happen.

Consider the function $V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$.

The equation $V(x, y, z) = v$ for $v > 0$ defines an ellipsoid centered at the point (0,0,2*r*). Call this ellipsoid E_1.

Proposition: There exists v^* such that any trajectory that starts outside the ellipsoid $V = v^*$ eventually enters this ellipsoid and remains trapped for all future time.

(We use an ellipse since it is easier to draw than an ellipsoid)



Trajectories Remain Bounded

Proposition: There exists v^* such that any trajectory that starts outside the ellipsoid $V = v^*$ eventually enters this ellipsoid and remains trapped for all future time.

Proof: Compute

$$\dot{V} = -2\sigma\left(rx^2 + y^2 + b(z^2 - 2rz)\right)$$

or

$$\dot{V} = -2\sigma(\underbrace{rx^2 + y^2 + b(z-r)^2} - br^2)$$

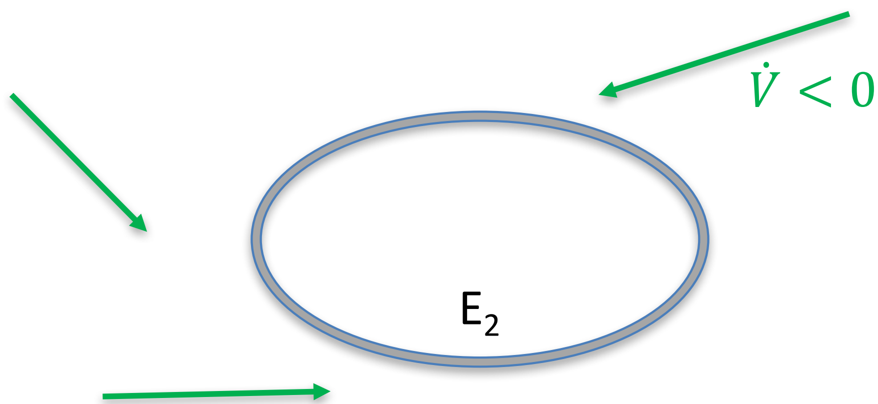
But

$$rx^2 + y^2 + b(z-r)^2 = \mu$$

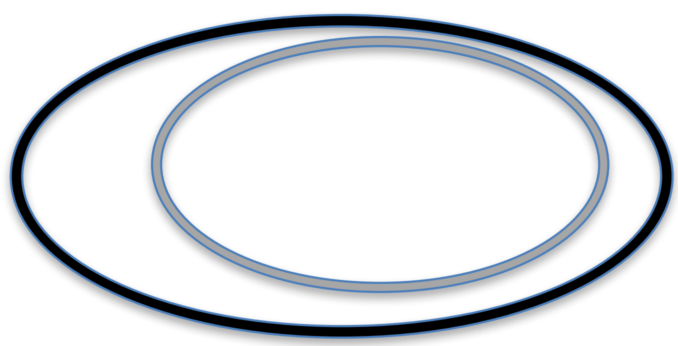
is itself an ellipsoid when $\mu > 0$, and when $\mu > br^2$ we have $\dot{V} < 0$.

Trajectories Remain Bounded

In particular, denote the ellipsoid $rx^2 + y^2 + b(z - r)^2 = br^2$ as E_2.

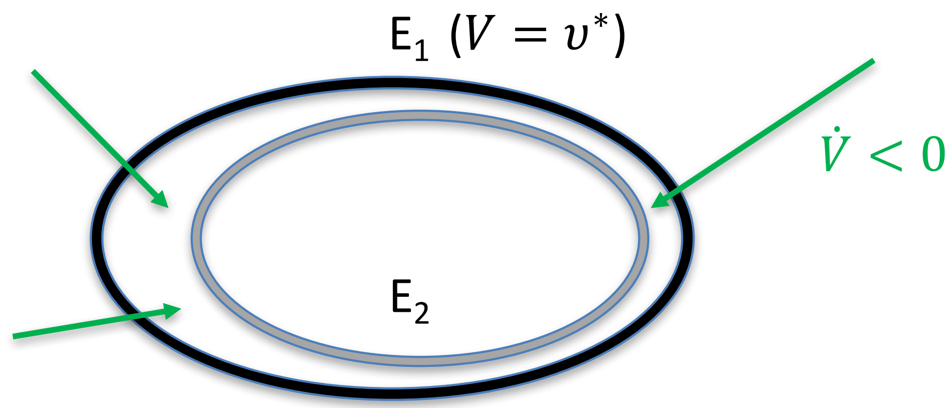


We can now pick a number v^* large enough so that ellipsoid E_1 contains ellipsoid E_2.



Trajectories Remain Bounded

Then $\dot{V} < 0$ for all $v > v^*$ and trajectories starting outside E_1 eventually enter it.

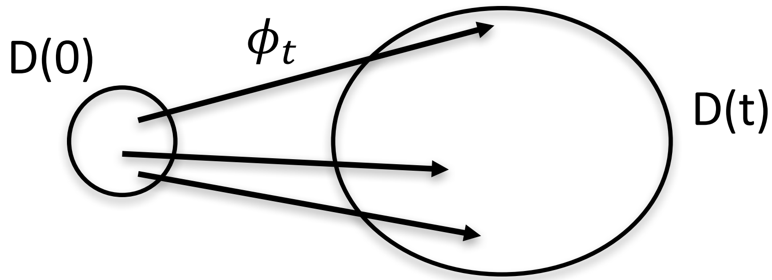


As a consequence, all trajectories starting far from the origin are attracted to a set that is contained within the ellipsoid $V(x, y, z) = v^*$, so **all trajectories are bounded.**

What is the Volume of the Attractor?

We have shown that for all $r > 0$ the trajectories remain bounded. So for $r > r_H$ there must be an attractor. *What is its volume?* In theory, it could be quite large.

Let $D(0)$ be a compact region in phase space. Then denote the trajectory from a point starting in $D(0)$ as ϕ_t. This is also called "the flow" from that point. The flow set at time *t* from points starting in $D(0)$ is then $D(t) = \phi_t(D)$.



Let $V(t)$ denote the volume of $D(t)$, and $\vec{F}(\vec{x})$ be the vector field for a 3-D system of ODEs. The divergence of this vector field is

$$\operatorname{div} \vec{F} = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}$$

It measures how fast volumes change under the flow ϕ_t of $\vec{F}(\vec{x})$.

What is the Volume of the Attractor?

There is a theorem, called Liouville's theorem, that relates the rate of change of volume in phase space to the divergence of the velocity vector field:

$$\frac{dV}{dt} = \int_{D(t)} \mathrm{div}\, \vec{F} dxdydz$$

where *V* is the volume of the flow set *D*.

For the Lorenz equations the divergence is constant: $\mathrm{div}\, \vec{F} = -(\sigma + 1 + b)$, so

$$\frac{dV}{dt} = -(\sigma + 1 + b)V$$

With solution

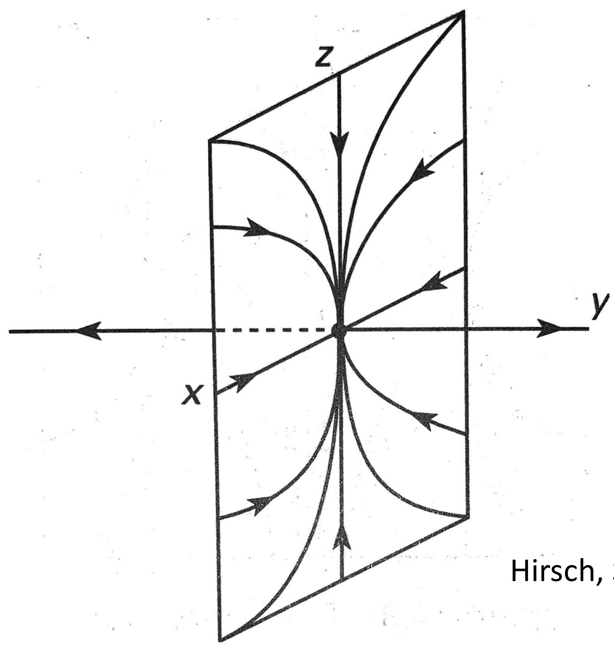
$$V(t) = V(0)e^{-(\sigma+1+b)t}$$

So trajectories move exponentially quickly to an attractor of volume 0.

What is the Attractor?

For $r < r_H$ the attractor is one or more equilibria. What about for $r > r_H$?

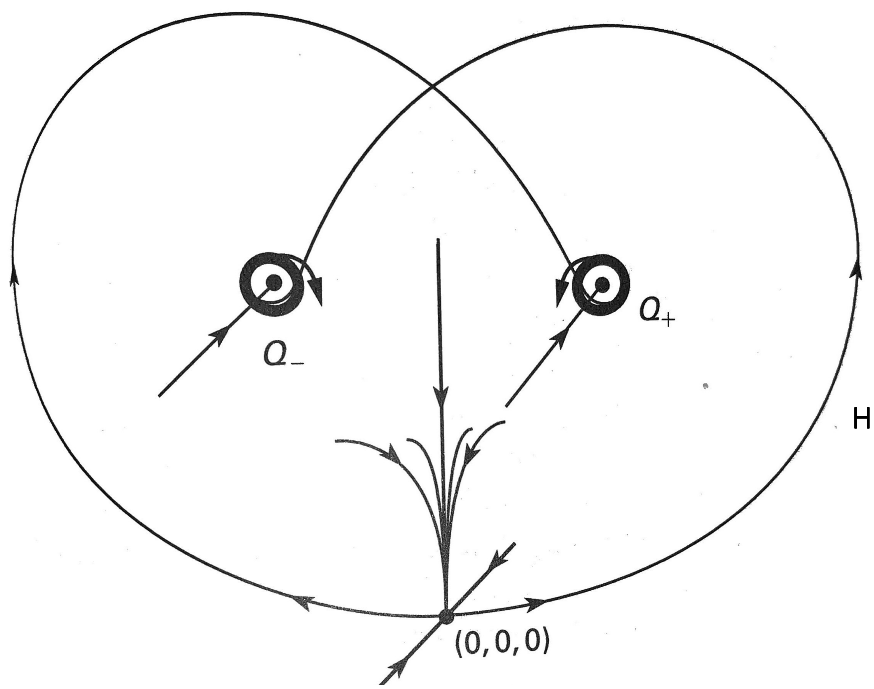
If you linearize about the equilibrium at the origin, you see that there are two real negative eigenvalues, and one real positive eigenvalue. The flow near the origin appears as below:



Hirsch, Smale, Devaney

What is the Attractor?

The figure below shows numerical solutions extending the two branches of the unstable manifold of the equilibrium at the origin.

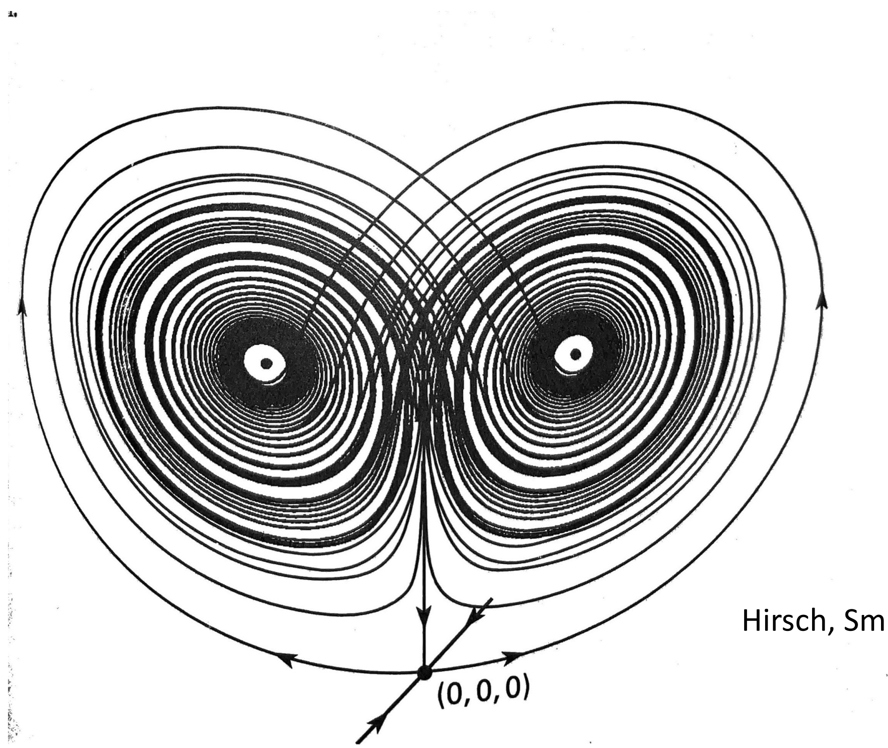


Hirsch, Smale, Devaney

The equilibria denoted as Q_+ and Q_- are what we denoted as C^+ and C^-, respectively.

A Strange Looking Attractor

If we do this again, but continue for longer times, we get the following:

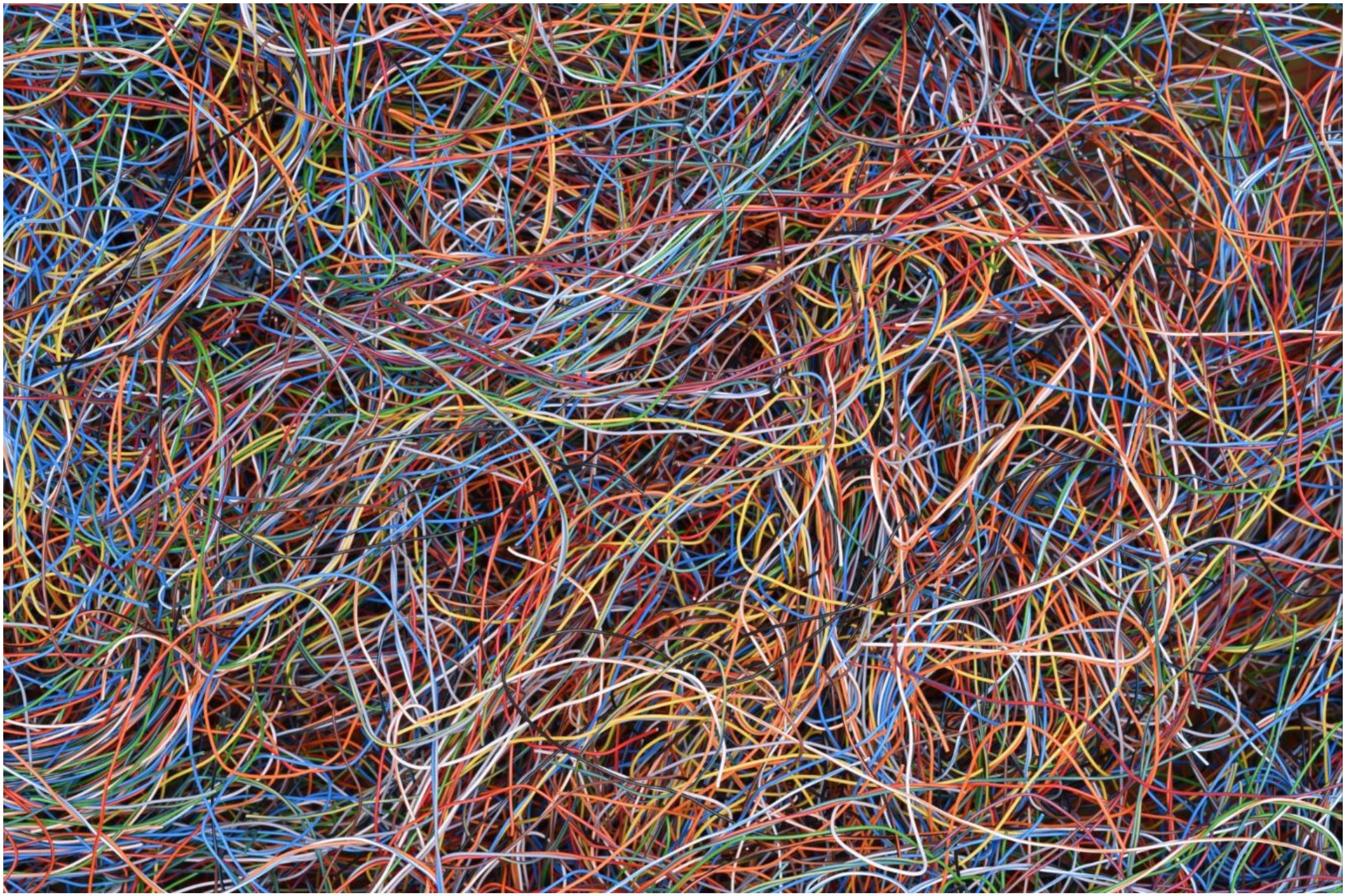


Hirsch, Smale, Devaney

This is called the Lorenz attractor. It is an example of what are now called strange attractors.

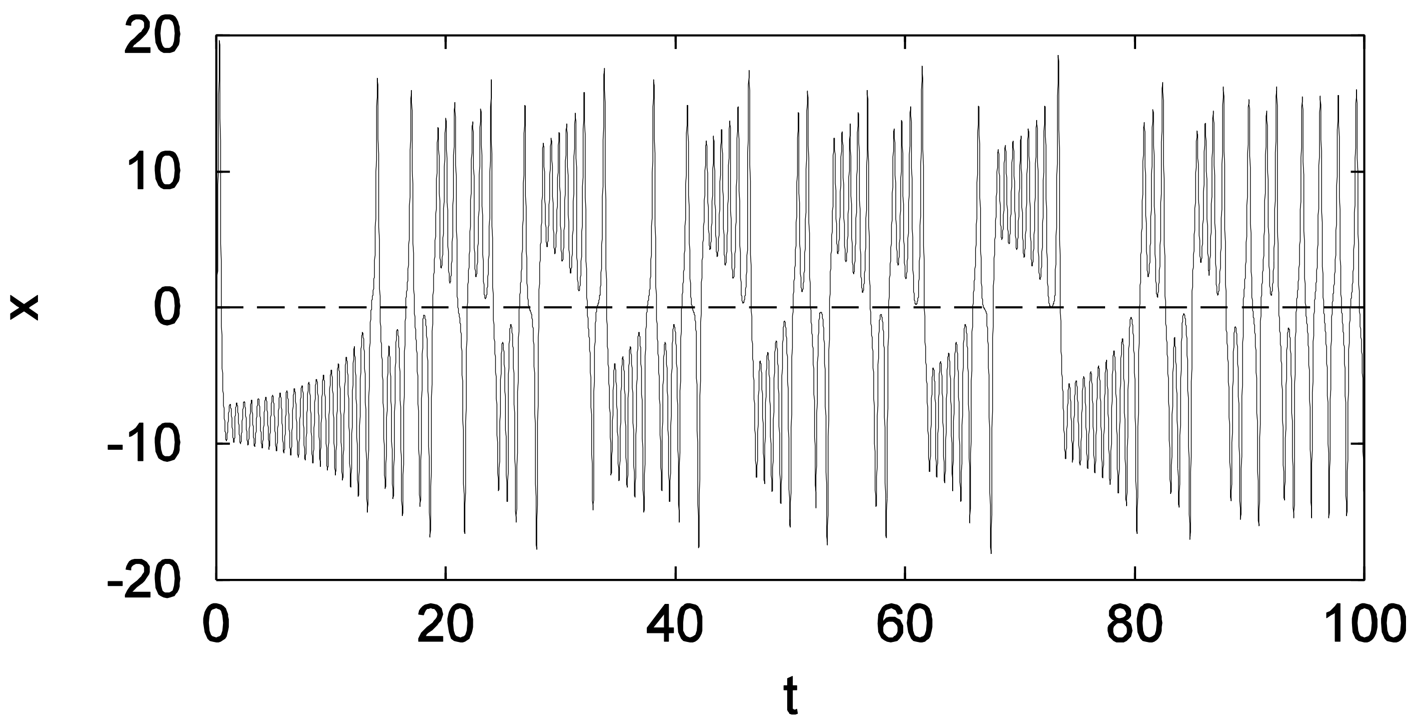
Animation of Flow on the Lorenz Attractor

Chaos



A Time Course on the Lorenz Attractor

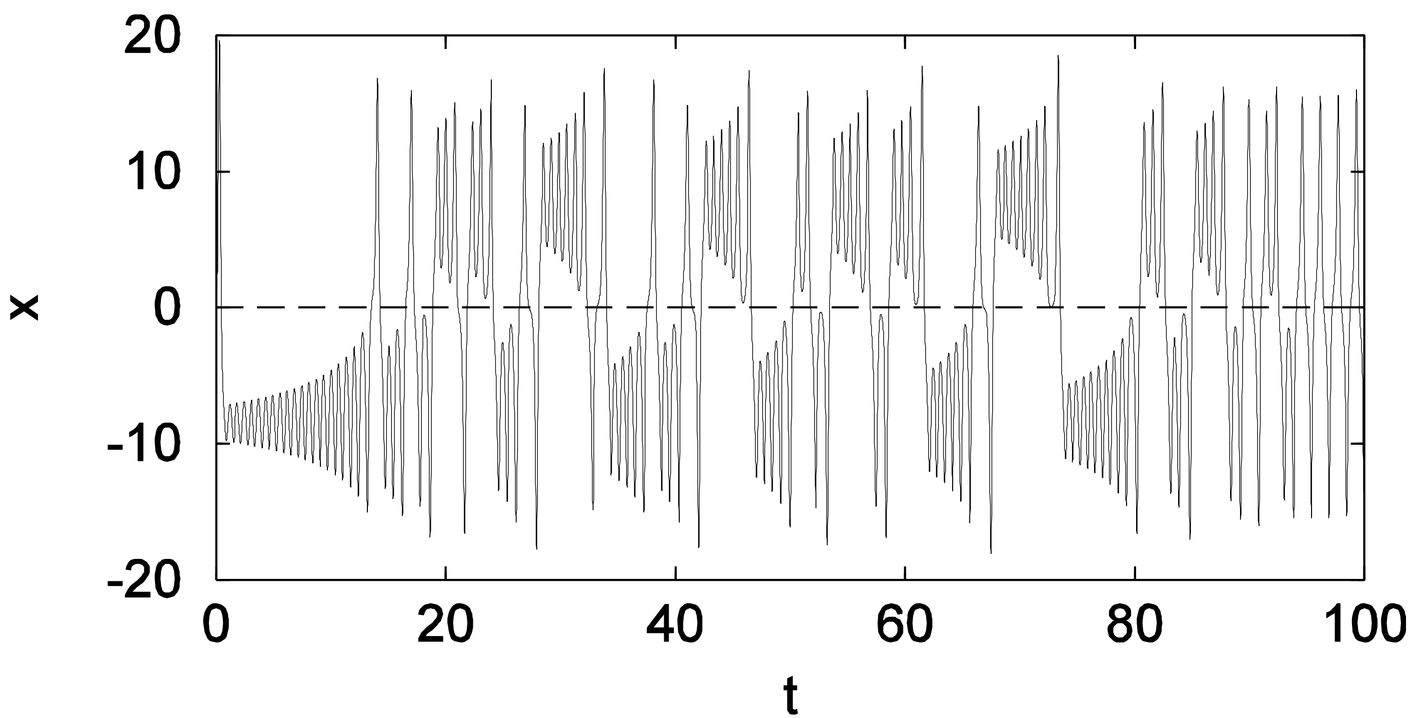
For a trajectory that is attracted to the Lorenz attractor, what does the time course look like?



There is little order, and it's impossible to predict at one point to the next whether *x* will be positive or negative. Positive *x* means the phase point is on one "wing" of the strange attractor, while negative *x* means it is on the other wing.

This Looks Chaotic

A time course like this is called chaotic, for obvious reasons.

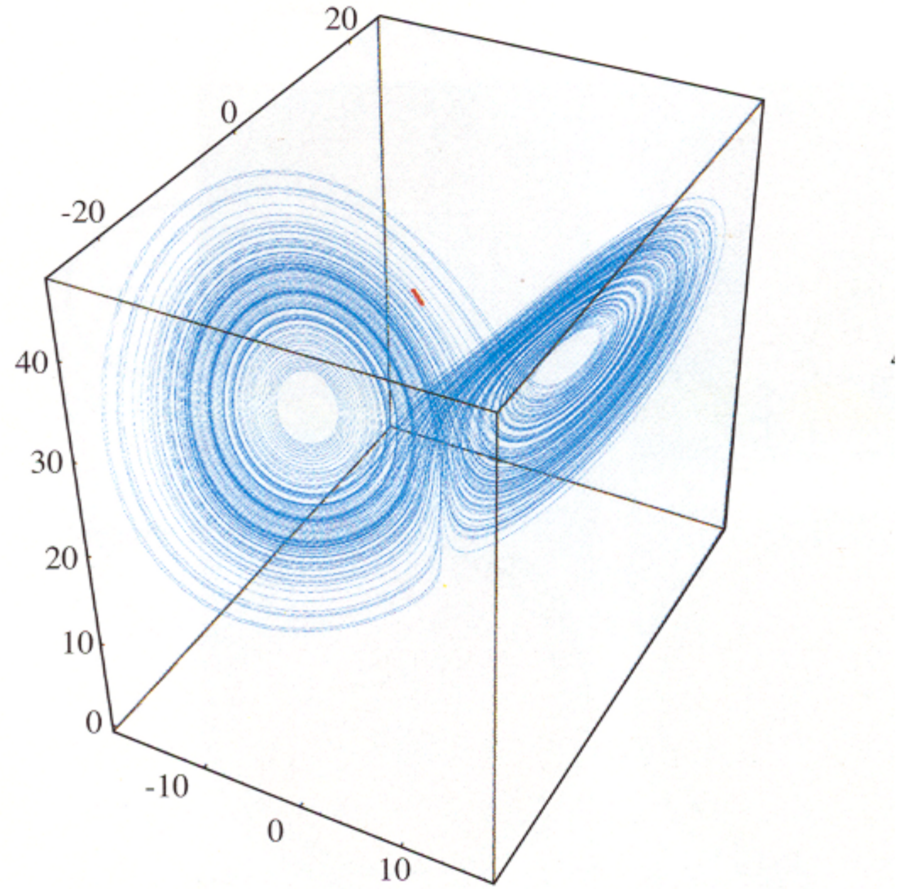


But how does one formally define chaos?

Follow the Blob

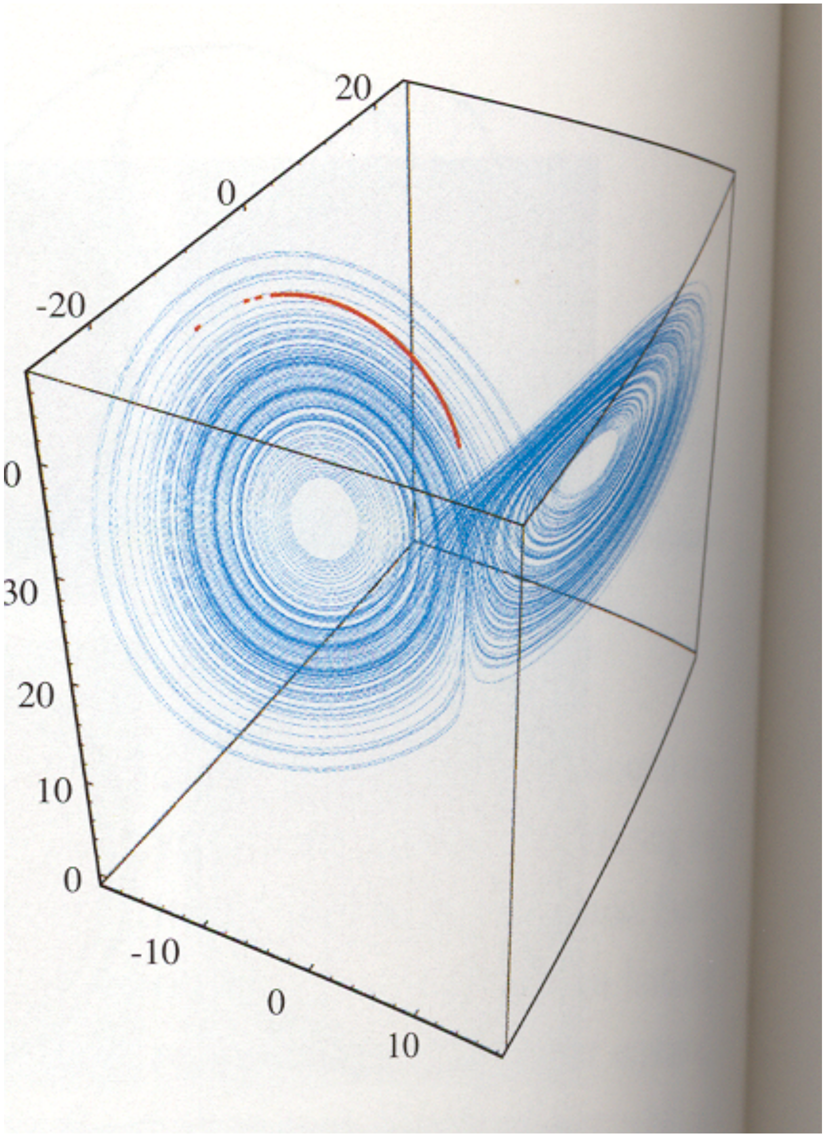
Suppose you start with a small blob of points on or near the Lorenz attractor in phase space. For attractors like equilibria or limit cycles, a blob of points near the attractor would contract as the trajectories moved towards the attractor. What happens to the blob on the Lorenz attractor?

Initial blob
(in red)



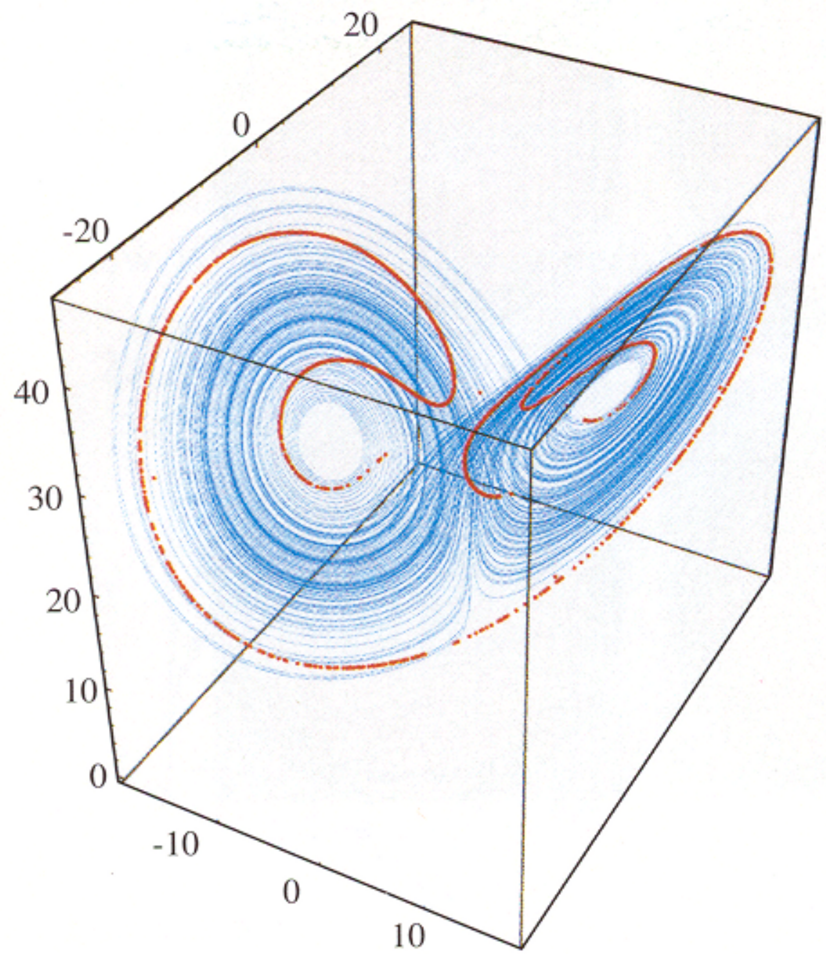
Follow the Blob

A little later in time



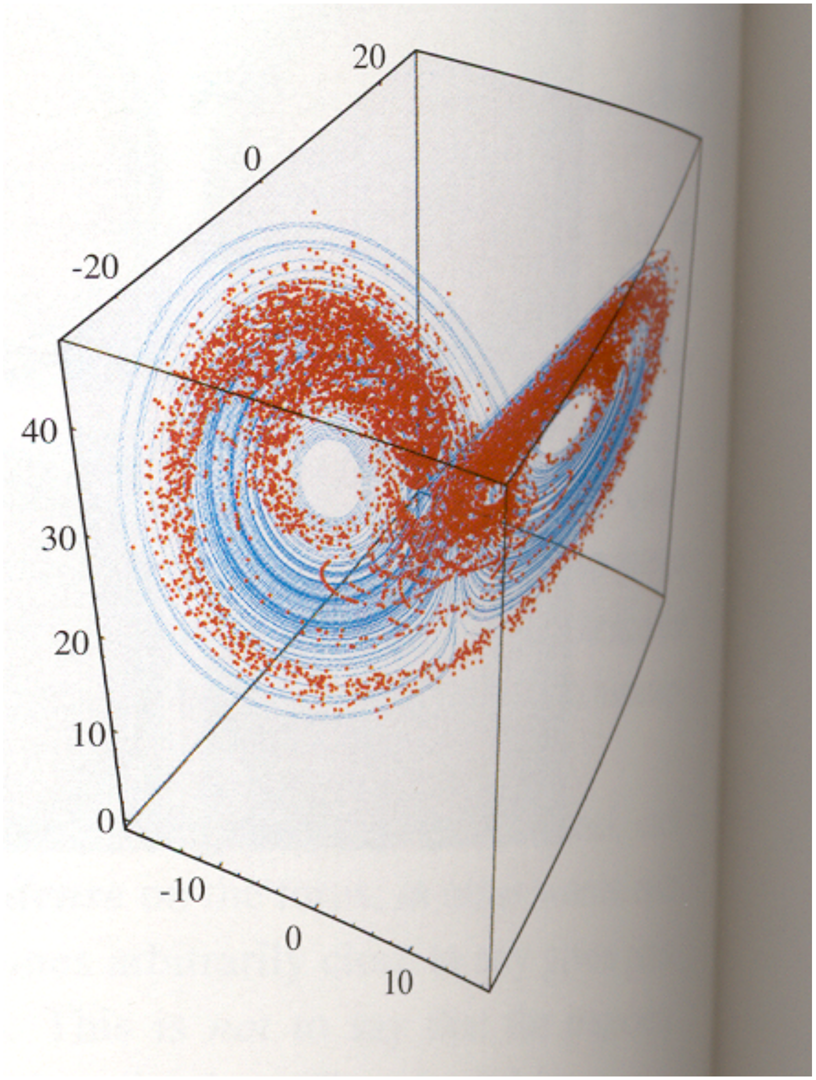
Follow the Blob

A little later in time



Follow the Blob

A little later in time

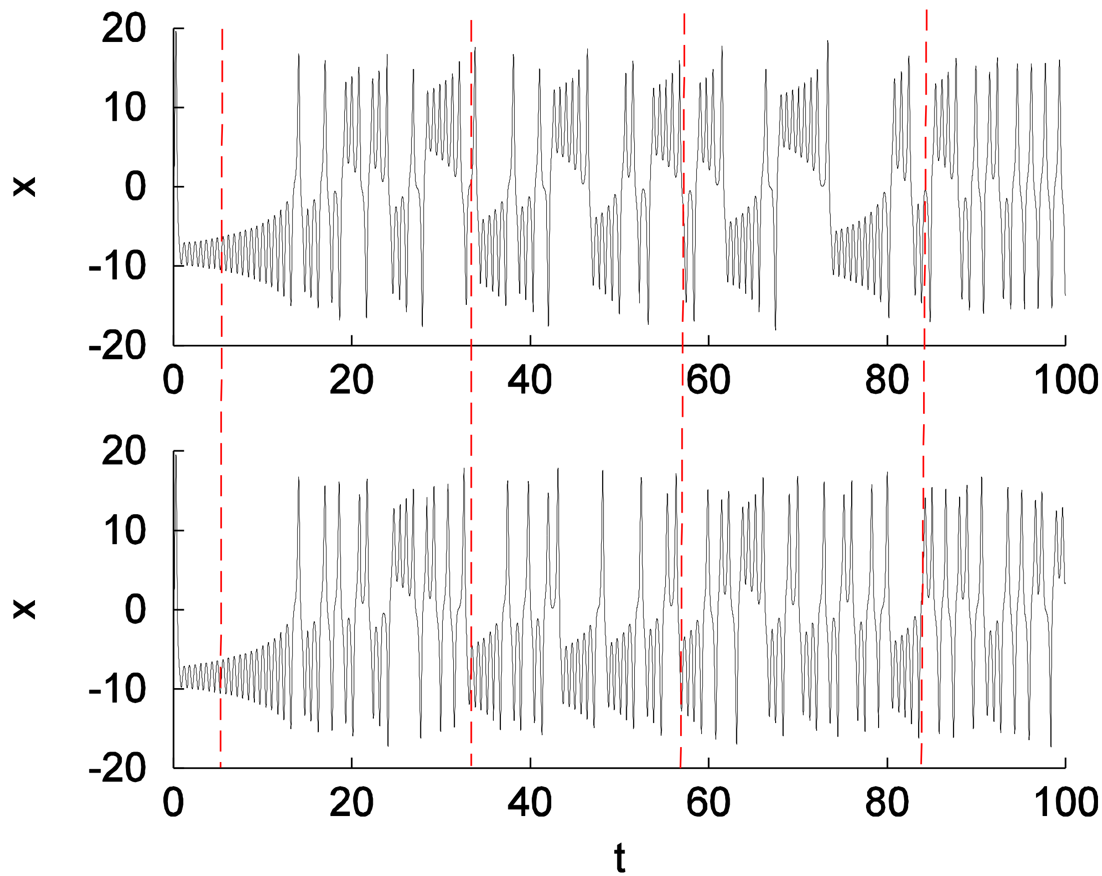


A Nice Video

https://www.youtube.com/watch?v=FYE4JKAXSfY

Definition of Chaos

A dynamical system is chaotic if it is **sensitive to initial conditions**

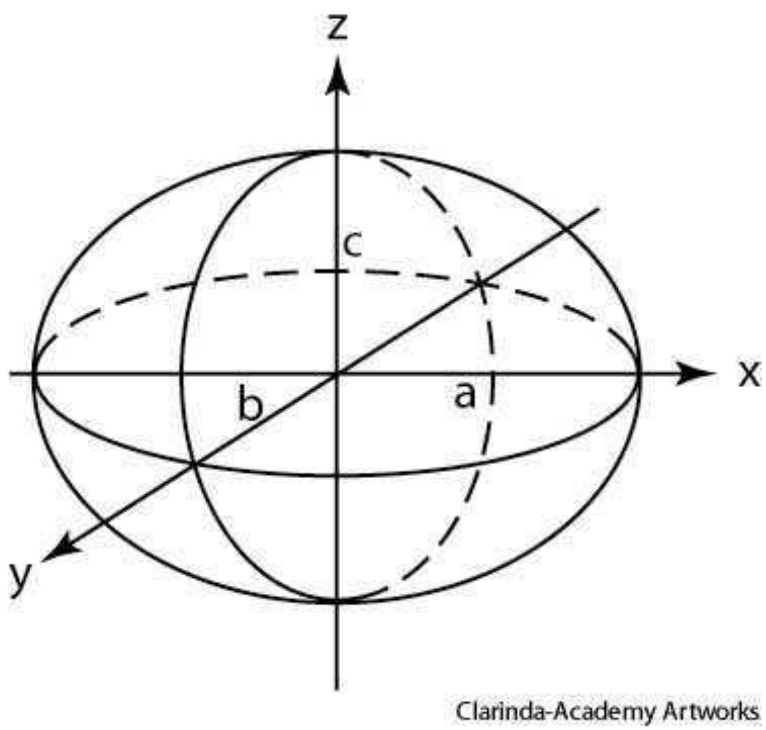


Two time courses starting from almost-identical initial conditions

This is why **prediction is impossible** for chaotic systems.

Local Lyapunov Spectrum

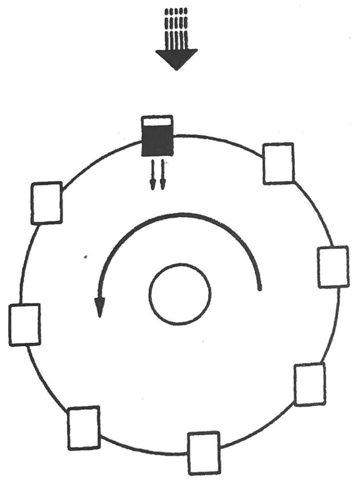
Typically, the expansion of the initial sphere is different along different axes. The sphere first deforms into an ellipsoid, with three axes, $\vec{a}$, $\vec{b}$, and $\vec{c}$. There will be a Lyapunov exponent associated with each axis, so λ_a, λ_b, and λ_c. The size of the axis increases or decreases proportional to $e^{\lambda_j t}$.



These are the Lyapunov spectrum, and for a chaotic system at least one $\lambda_j > 0$.

The Chaotic Waterwheel

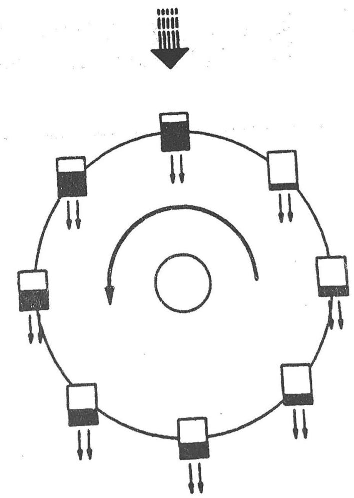
In the 1970s Lou Howard and William Malkus came up with a mechanical model of the Lorenz equations. It consists of a toy waterwheel with leaky paper cups and a water source at the top.



Lou Howard

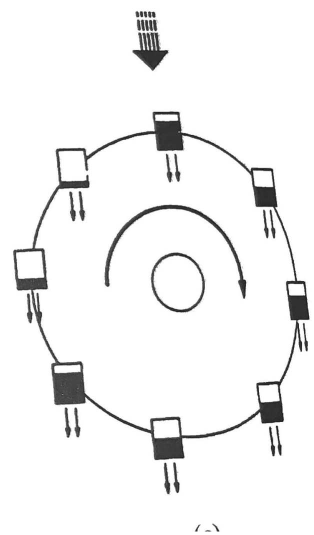
At low flow rates the wheel turns slowly in one direction or the other.

The Chaotic Waterwheel



At higher flow rates the wheel turns faster, but in one direction or the other.

The Chaotic Waterwheel

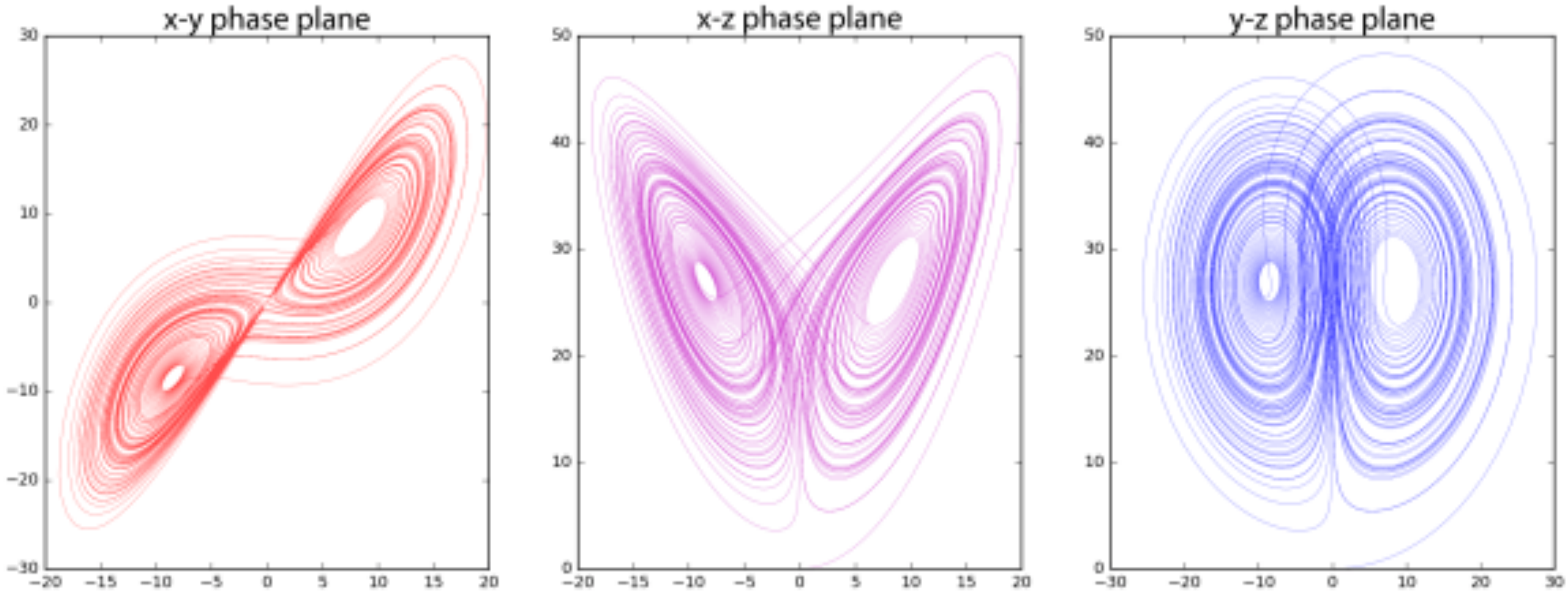


At even higher flow rates the wheel turns one way, and then the other, since the leak out of the cups is too slow to compensate for the inflow. Heavy, water-filled cups oppose the spinning of the wheel and make it spin the other way. **The spins reverse orientation in a chaotic manner when the flow is large enough.**

The Chaotic Waterwheel

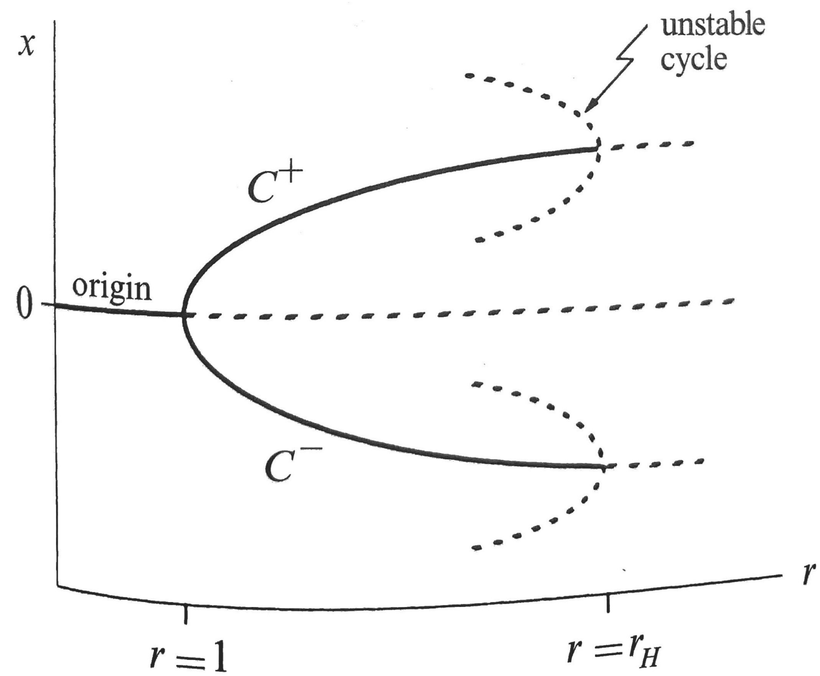
https://www.youtube.com/watch?v=FmhKN1Hx7z4

Bifurcation Analysis of the Lorenz System



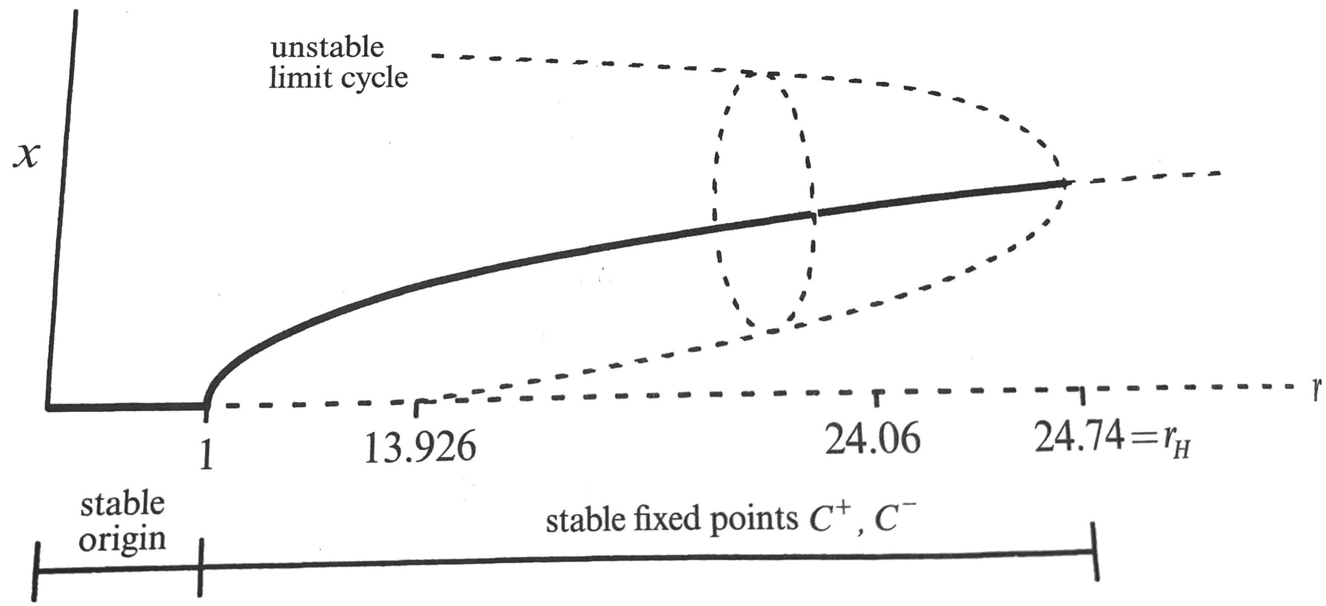
Analysis for Small Values of *r*

As discussed previously, the Lorenz system (with $\sigma = 10$ and $b = \frac{8}{3}$) has a stable equilibrium at the origin for $r < 1$. There is a pitchfork bifurcation at $r = 1$, and a Hopf bifurcation at $r \approx 24.74$. The bifurcation diagram looks like the following:



Analysis for Small Values of *r*

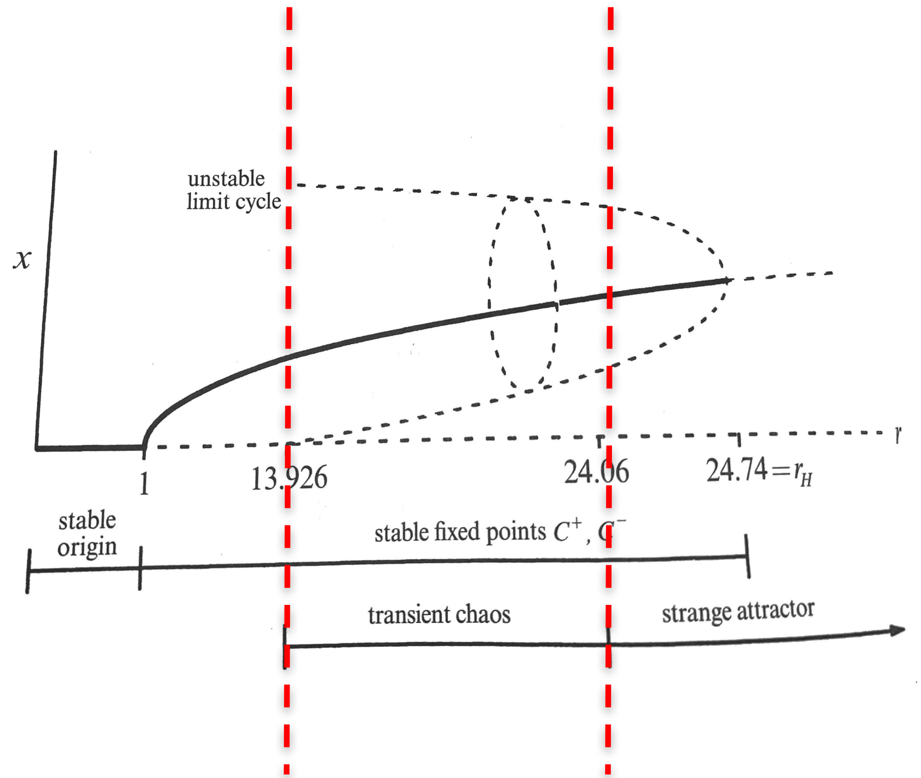
What happens to the branches of unstable limit cycles? Are there saddle-node of periodic bifurcations, where the limit cycles turn around and become stable?



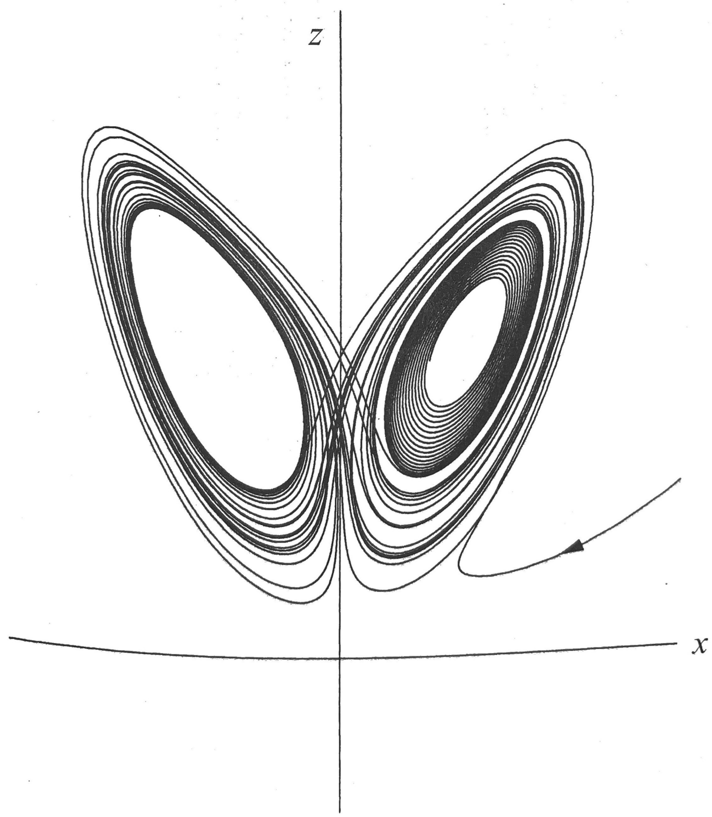
No, the unstable periodic branch terminates at a homoclinic bifurcation at $r \approx 13.926$.

Transient Chaos

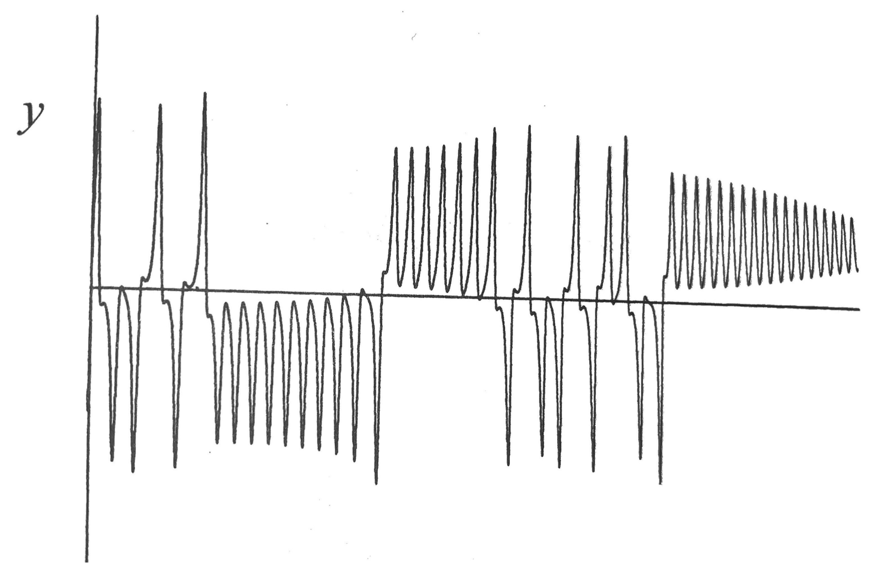
In the parameter interval between the homoclinic and the Hopf bifurcations the system exhibits transient chaos.



Transient Chaos



The trajectory starts out chaotic, but eventually settles onto the stable equilibrium.

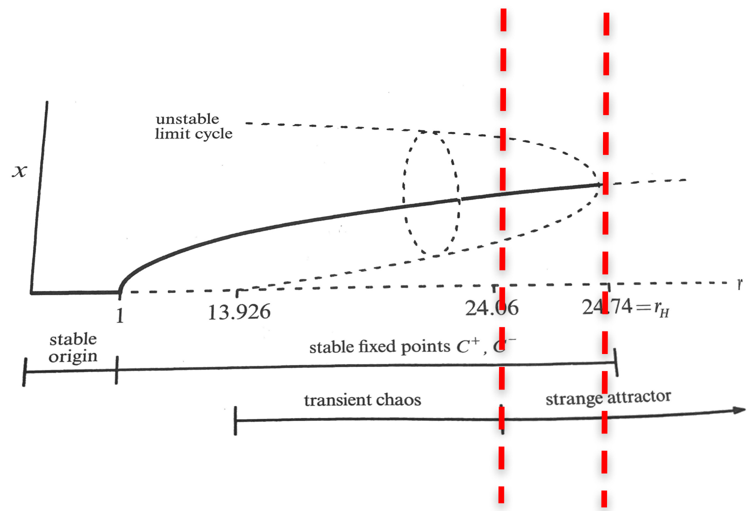


The chaotic wandering lasts longer for larger values of r.

Once $r \approx 24.06$ the chaos lasts forever

Tristability and Intermittency

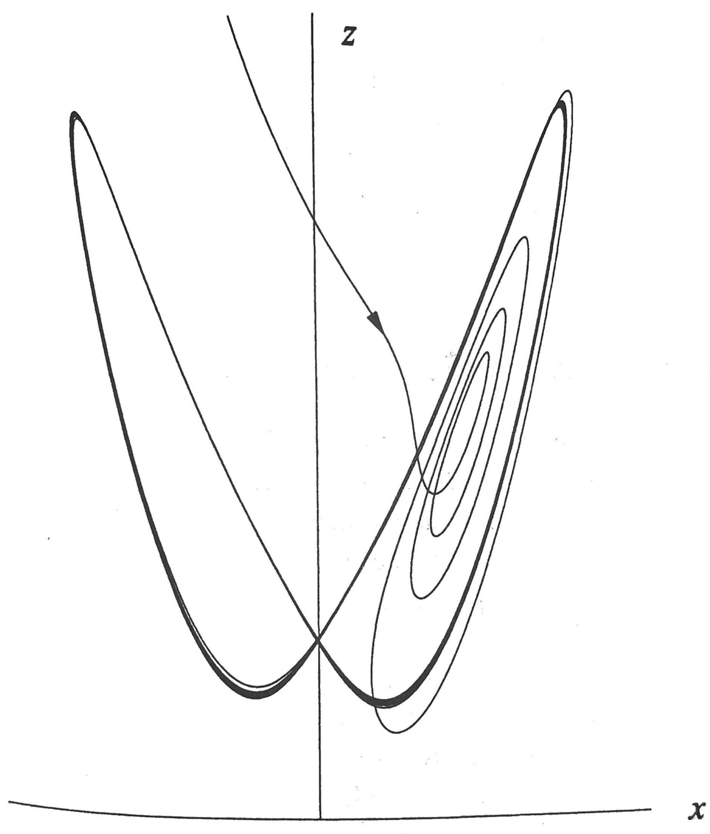
For $24.06 < r < 24.74$ there is tristability between the strange attractor and the equilibria C^+ and C^-.

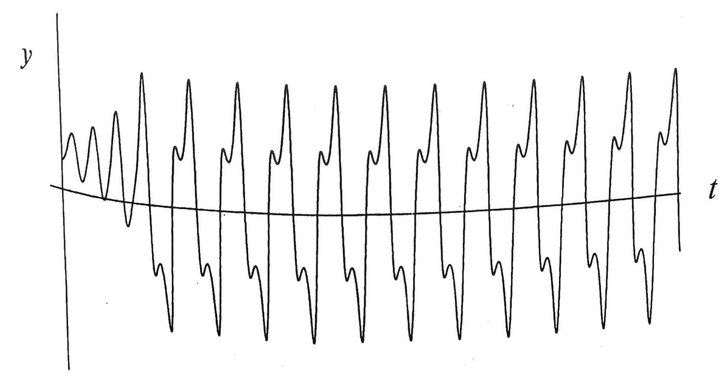


If a little noise were added to the system, the trajectory could follow the strange attractor for a while, then move to an equilibrium, then back to the strange attractor. This occasional chaotic activity is called intermittency.

Behavior for Larger *r* Values

For $r > r_H$ (24.47), the system is mostly chaotic. However, there are narrow windows of periodic behavior. For example, at $r = 350$,





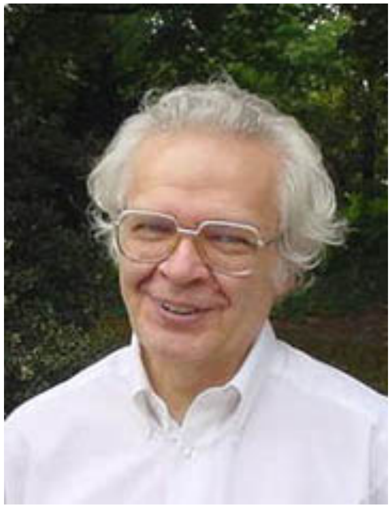
The Rössler System

The Lorenz system is only one example of a system of nonlinear ODEs that exhibits chaos. There are many others. A well-known example is the system studied by Otto Rössler in a 1976 paper, now called the Rössler system:

$$\frac{dx}{dt} = -y - z$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c)$$



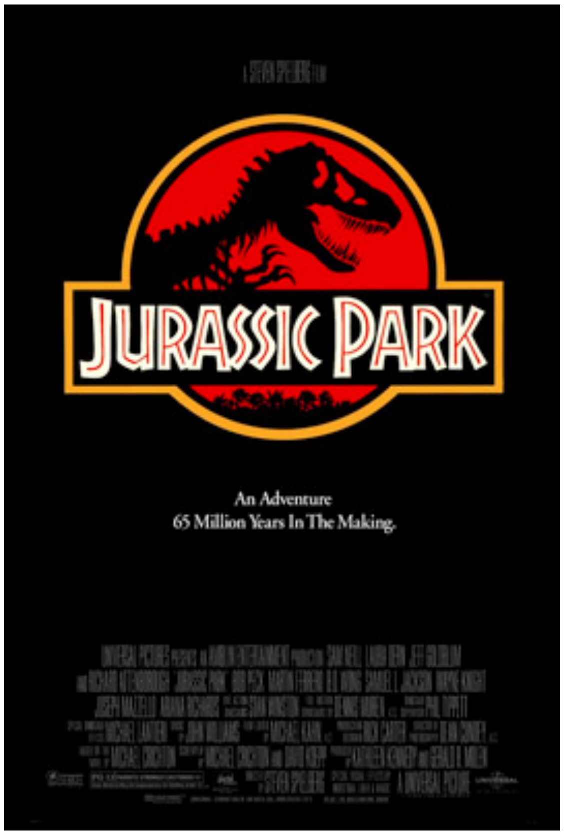
There is only one nonlinear term, but that is all that's needed for chaos when $a = 0.21$, $b = 0.2$, and $c = 5.5$.

Rössler Attractor Video

https://www.youtube.com/watch?v=abr9VhLIsJ4

Lorenz and Chaos Go To Hollywood

Jeff Goldblum, “chaotician”



Based on the novel by Michael Crichton

Chaos in Nonlinear Difference Equations

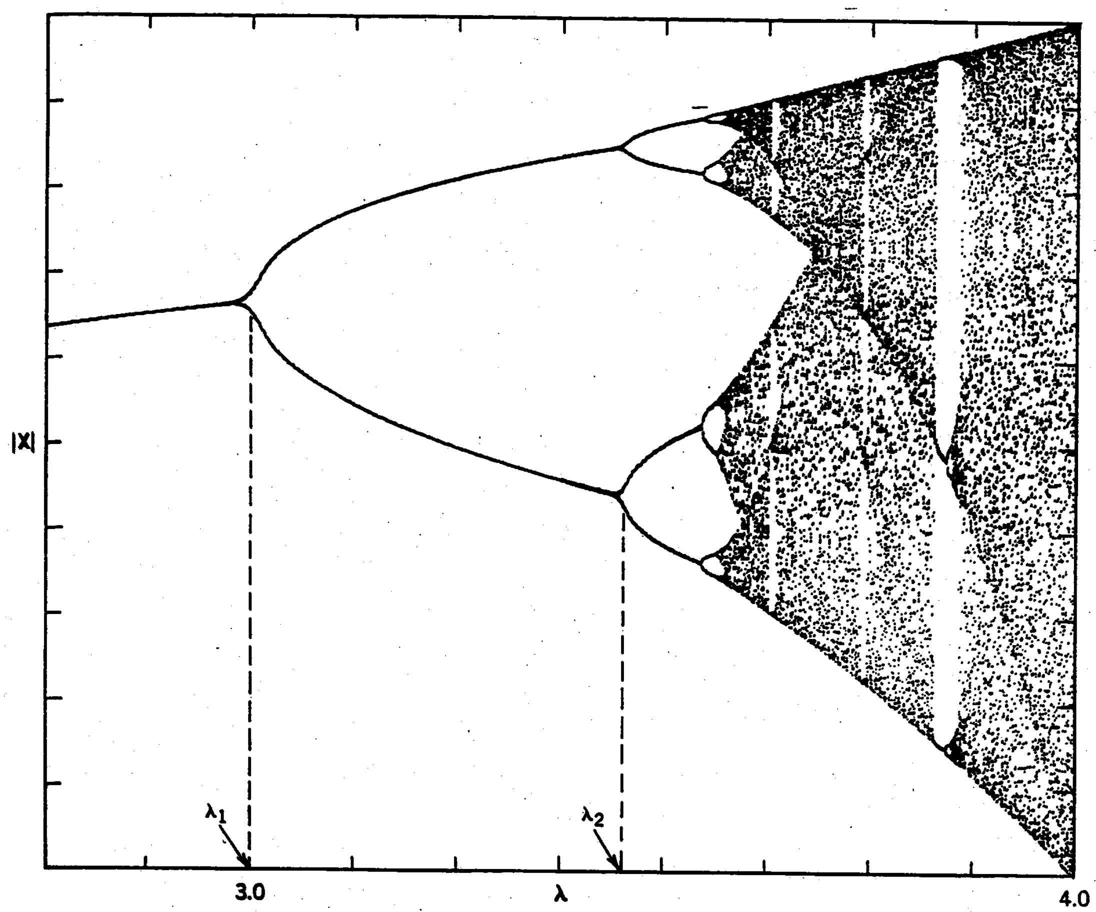


Figure 5-10 Bifurcation diagram for the quadratic map (5-3.3). Steady-state behavior as a function of the control parameter showing period-doubling phenomenon.

Chaos Can Occur in a Single Difference Equation

The dynamics of difference equations can easily become complex if there is a nonlinearity. The most well-studied example is the logistic equation for population dynamics, which has a quadratic nonlinearity

$$x_{n+1} = r x_n (1 - x_n)$$

and a single parameter $r > 0$.

We showed early in the semester that there are two equilibria: $x_1^* = 0$ and $x_2^* = \frac{r-1}{r}$.

<u>Case 1: $r < 1$</u>

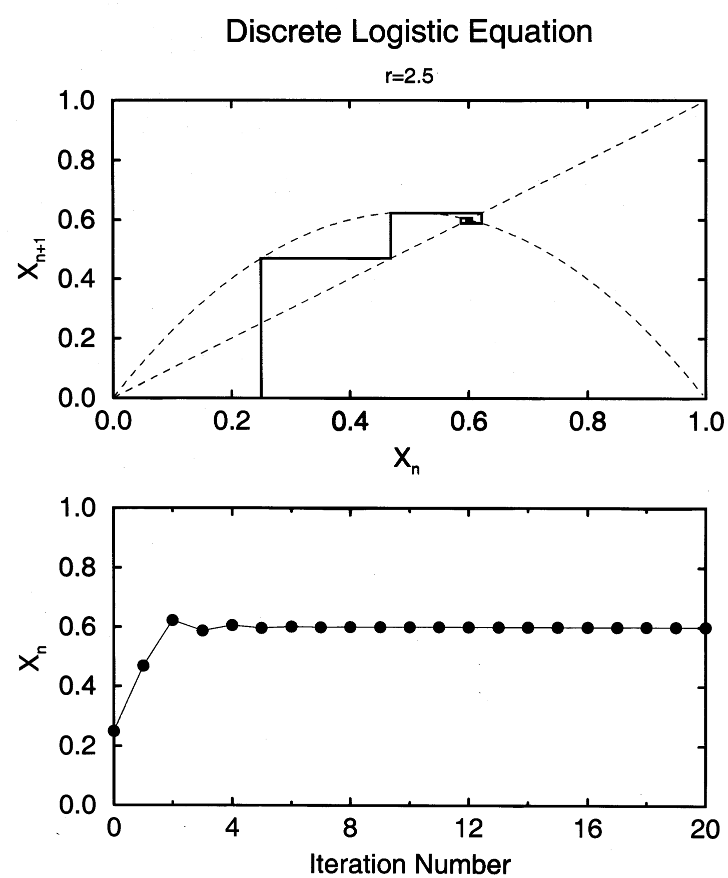
x_1^* is stable, x_2^* is unstable (and negative, so non-physical): the population eventually becomes extinct.

<u>Case 2: $1 \leq r < 3$</u>

There is a transcritical bifurcation at $r = 1$, at which point the stability switches. For $r < 3$ the equilibrium x_2^* is stable.

Population Persistence in the Logistic Equation

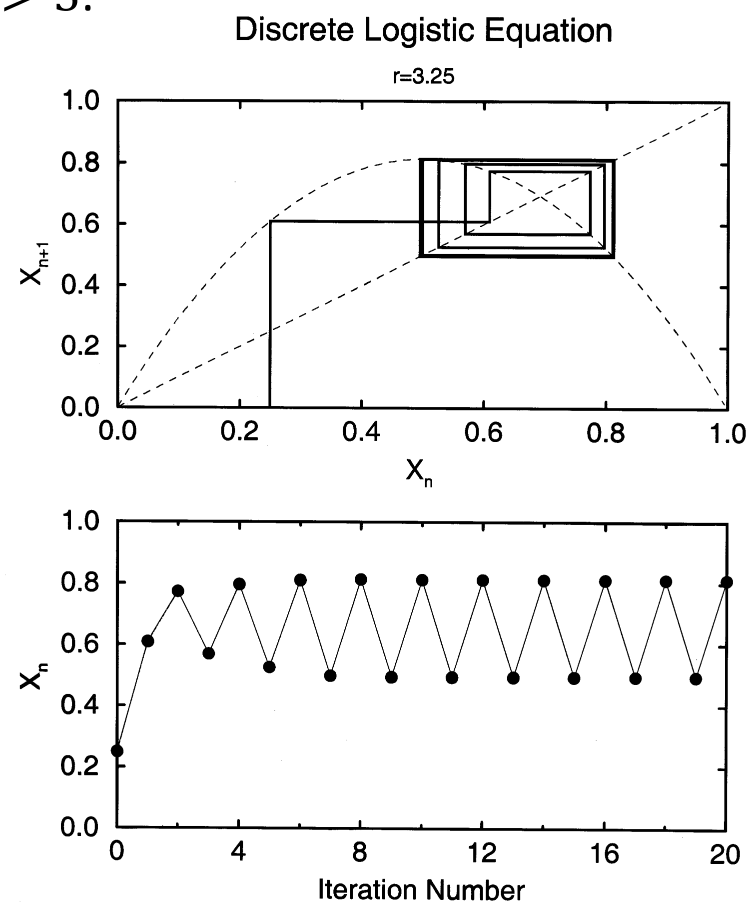
Cobweb diagram shows rapid convergence to the equilibrium x_2^*.



In terms of population biology, this means the population persists, it does not go extinct as it did with the lower reproduction rate $r < 1$.

Period Doubling or Flip Bifurcation

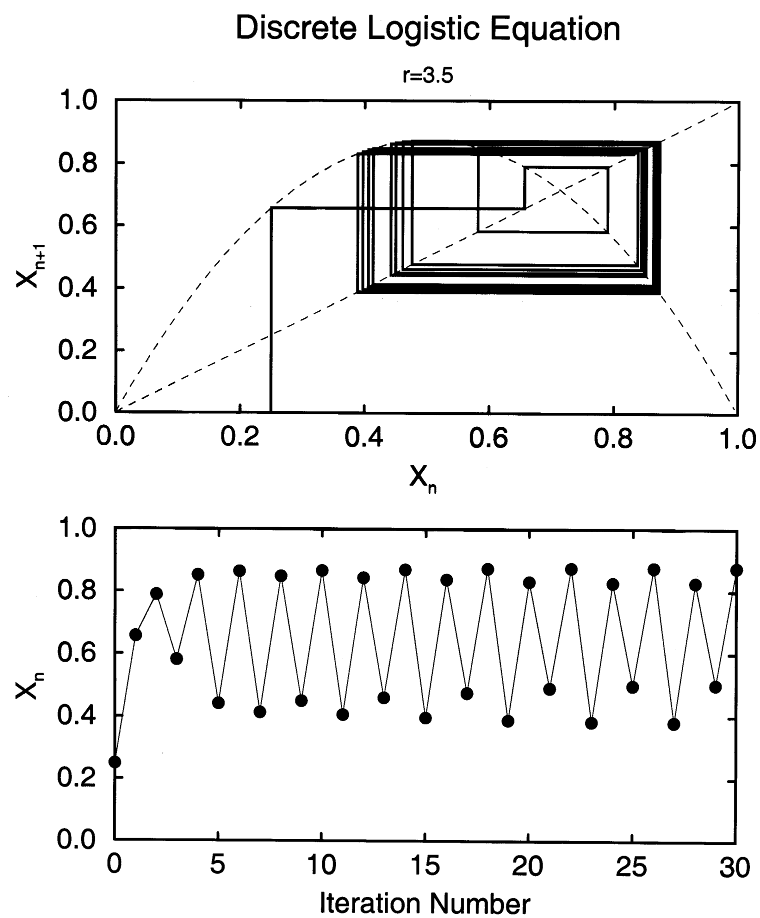
At reproduction rate $r = 3$, the equilibrium x_2^* loses stability at a flip or period doubling bifurcation, creating a stable 2-cycle that persists for a range of reproduction rates $r > 3$.



The population size now alternates between two values, year to year. A stable equilibrium population size should not be expected.

Cascade of Period Doublings

For larger *r* values, the period goes through an infinite sequence of period doublings:



There is now a four-cycle in the population size. The size returns to where it started every 4 iterations (which may mean every 4 years in an animal population).

Cascade of Period Doublings

Let r_n denote the value of r where a 2^n cycle first appears. Then, from computer simulations:

$r_1 = 3$ ⟶ 2 cycle

$r_2 = 3.449$ ⟶ 4 cycle

$r_3 = 3.54409$ ⟶ 8 cycle

$r_4 = 3.5644$ ⟶ 16 cycle

$r_5 = 3.568759$ ⟶ 32 cycle

$r_\infty = 3.568759 \ldots$ ⟶ period approaches ∞

This is called the u-sequence.

Cascade of Period Doublings

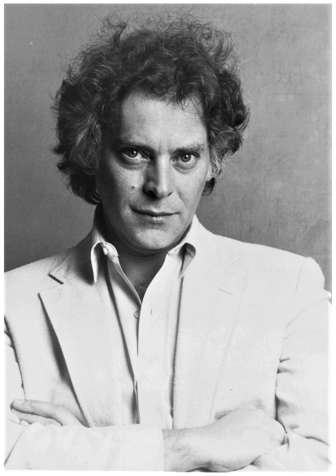
The successive bifurcations as r is increased come faster and faster. In the limit of large *n,* the distance between successive transitions shrinks by a constant factor

$$\boxed{\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669 \dots}$$

The u-sequence of period doubling bifurcations is a feature seen in all unimodel maps, not just the quadratic logistic map.

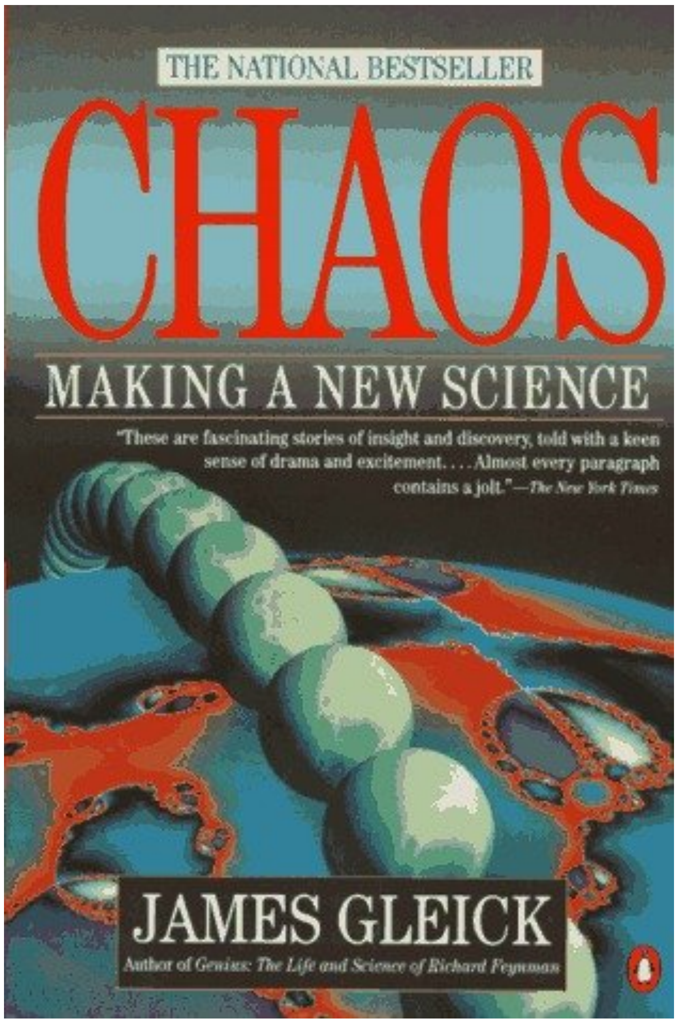
This was demonstrated by physicist Mitchell Feigenbaum in 1978 and 1979. It is an example of universality, and δ is a universal constant, now called the Feigenbaum constant. (Feigenbaum has a second universal constant named after him.)

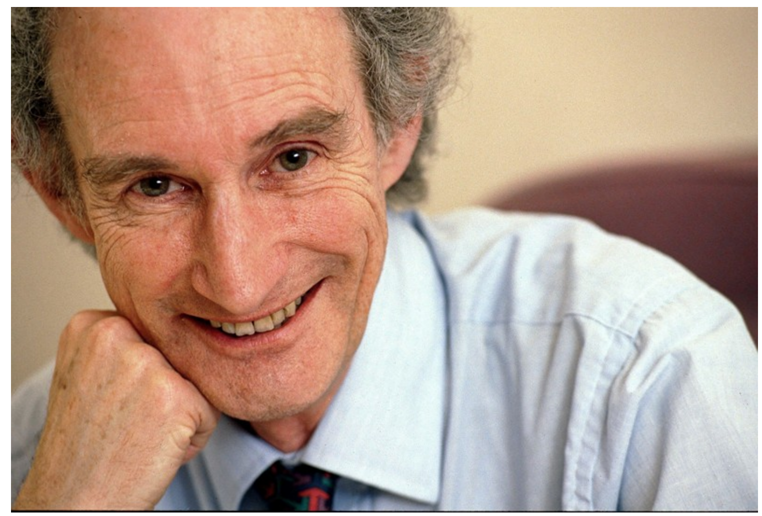
Mitchell Feigenbaum
1944-2019



Life After r_∞

According to biographer James Gleich, mathematician Robert May wrote the logistic equation in the hall as a problem for his graduate students to tackle, and asked "What the Christ happens for $r > r_\infty$?"





Robert May
(1936-2020)

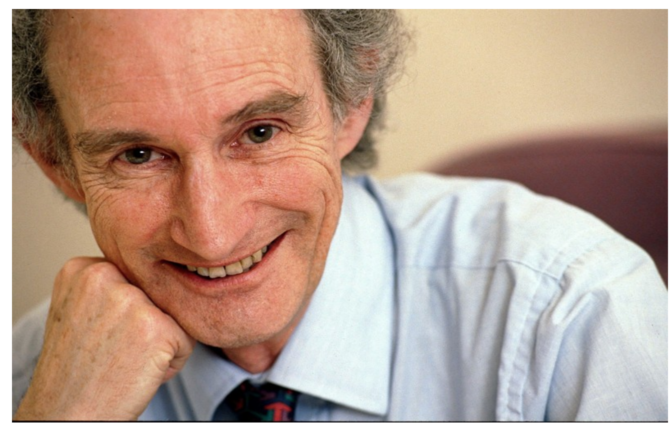
Life After r_∞

Nature Vol. 261 June 10 1976 **459**

review article

Simple mathematical models with very complicated dynamics

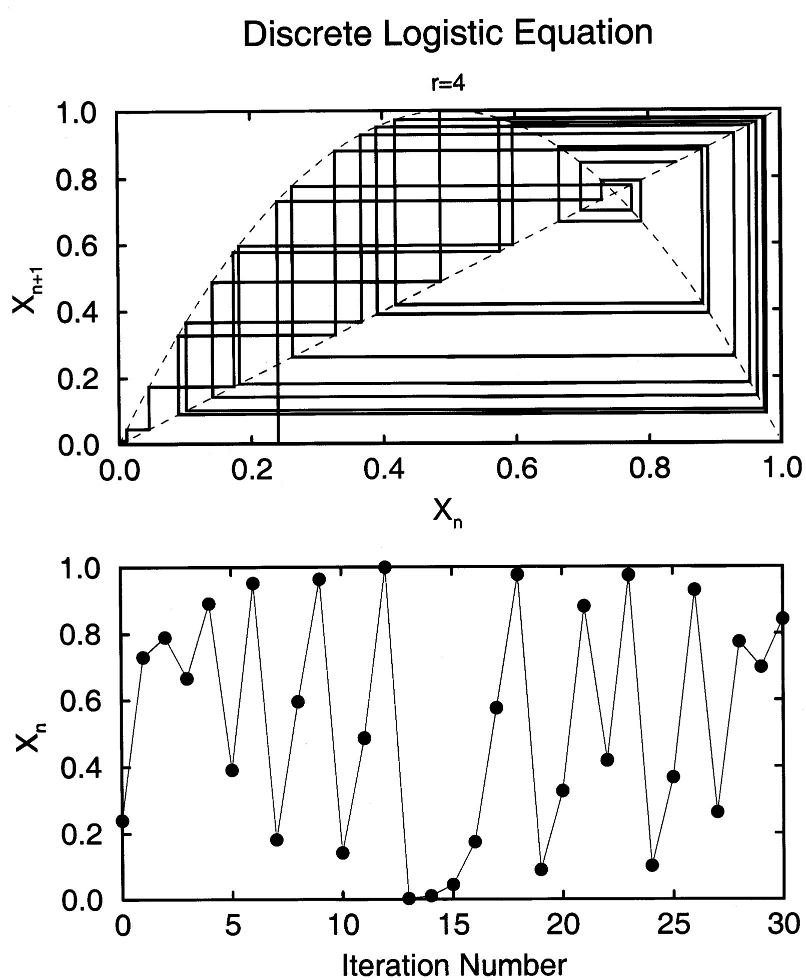
Robert M. May*



Robert May
(1936-2020)

Life After r_∞

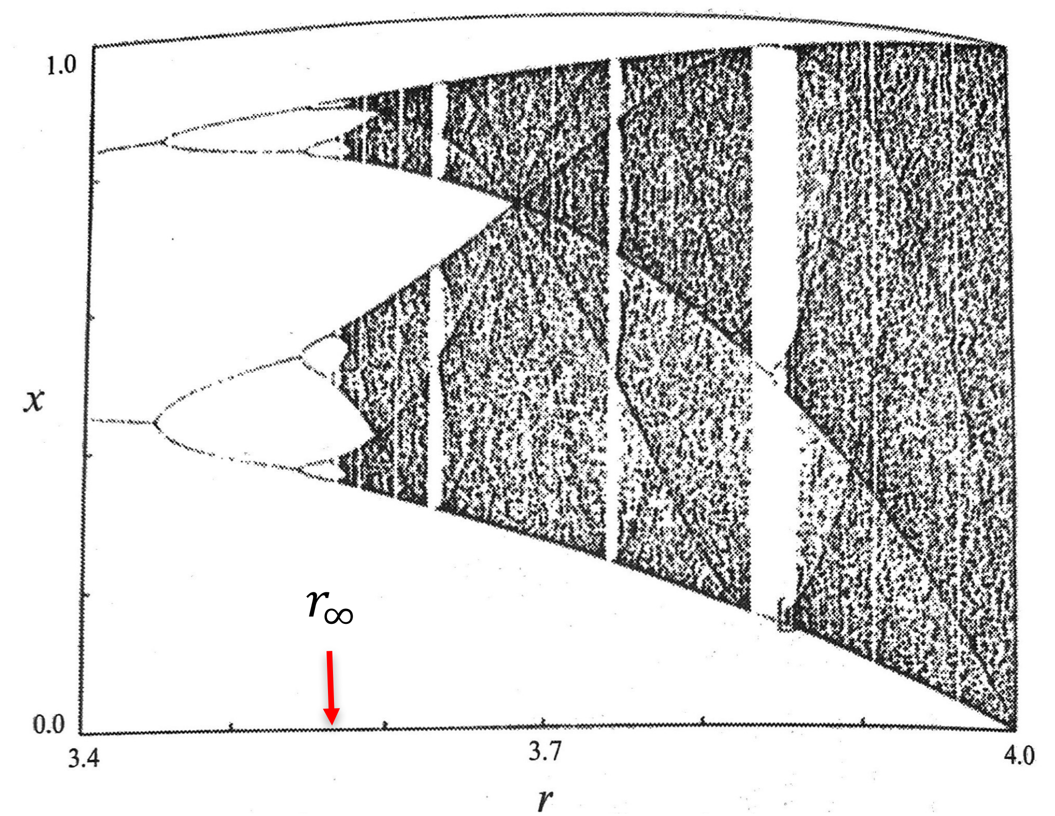
For this value of *r* the orbit is non-periodic. This is chaos!



A View From the Orbit Diagram

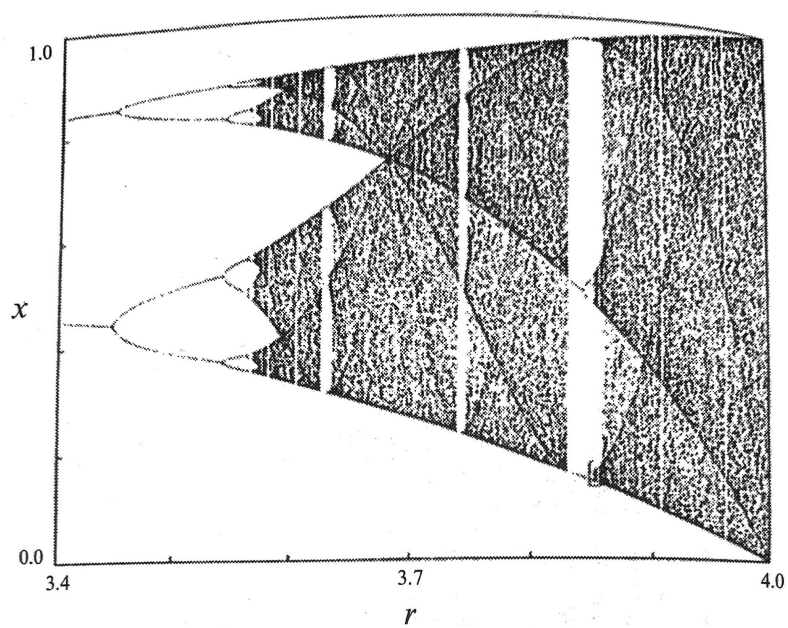
The orbit diagram shows the asymptotic values of x_n over a range of *r* values.

It has become iconic in the field of nonlinear dynamics.

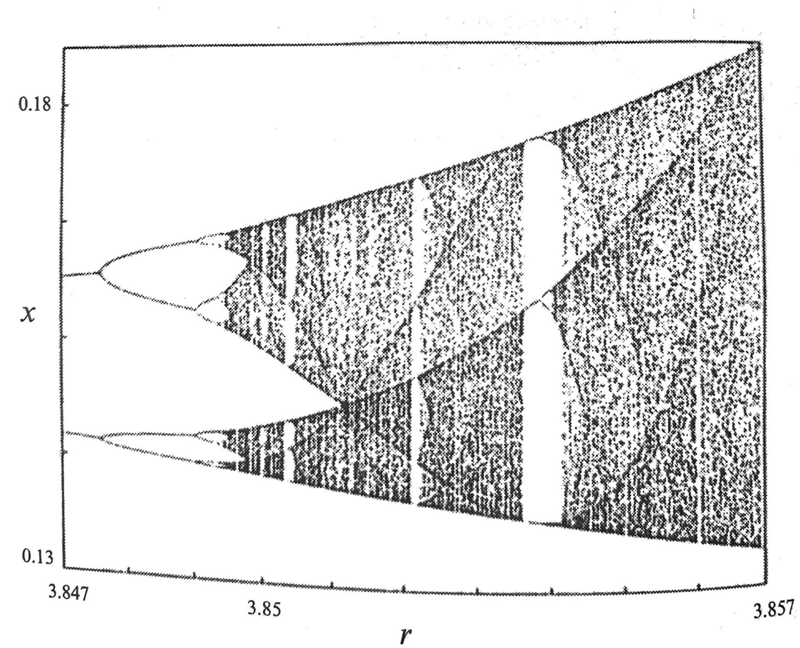


Self-Similar Structure

The orbit diagram exhibits a self-similar structure, which can be seen by focusing in on a very narrow interval of parameter space. This self-similarity is the hallmark of fractals.

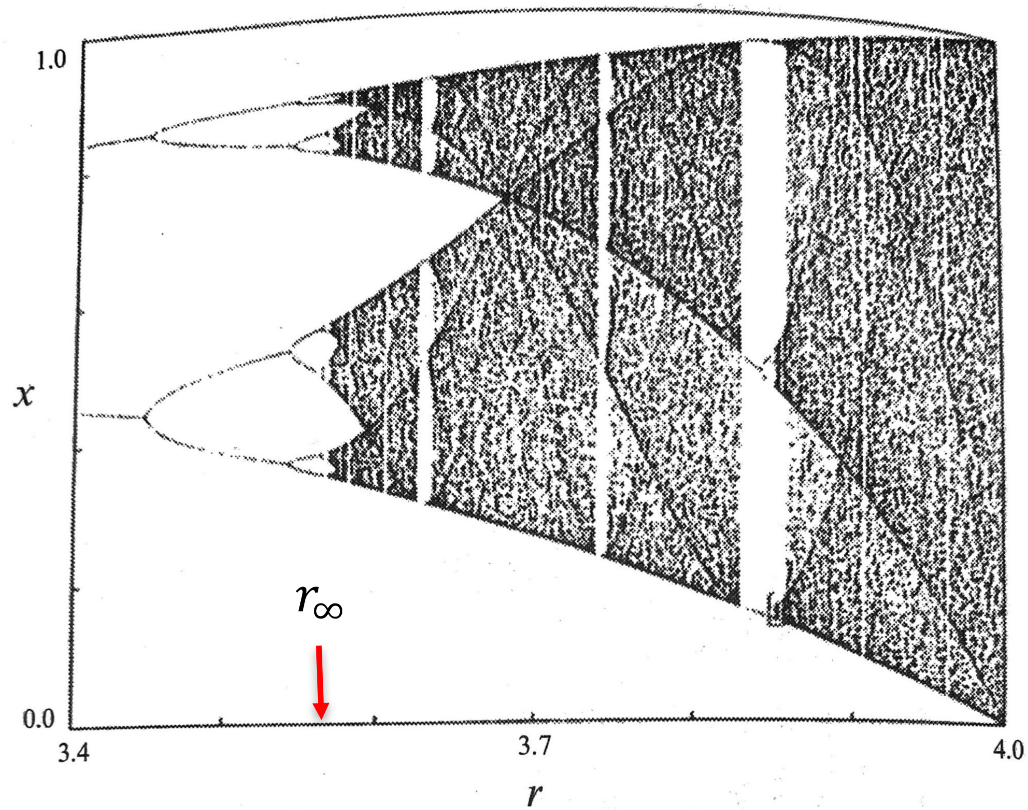


Interval from 3.4 to 4



Interval of 3.847 to 3.857

Not All is Chaos

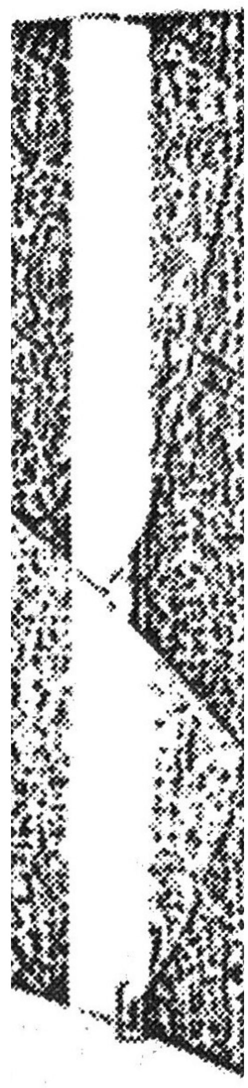


Notice that the complexity of the system does not keep growing as r is increased past r_∞. There are periodic windows of periodic behavior.

Period Three Implies Chaos

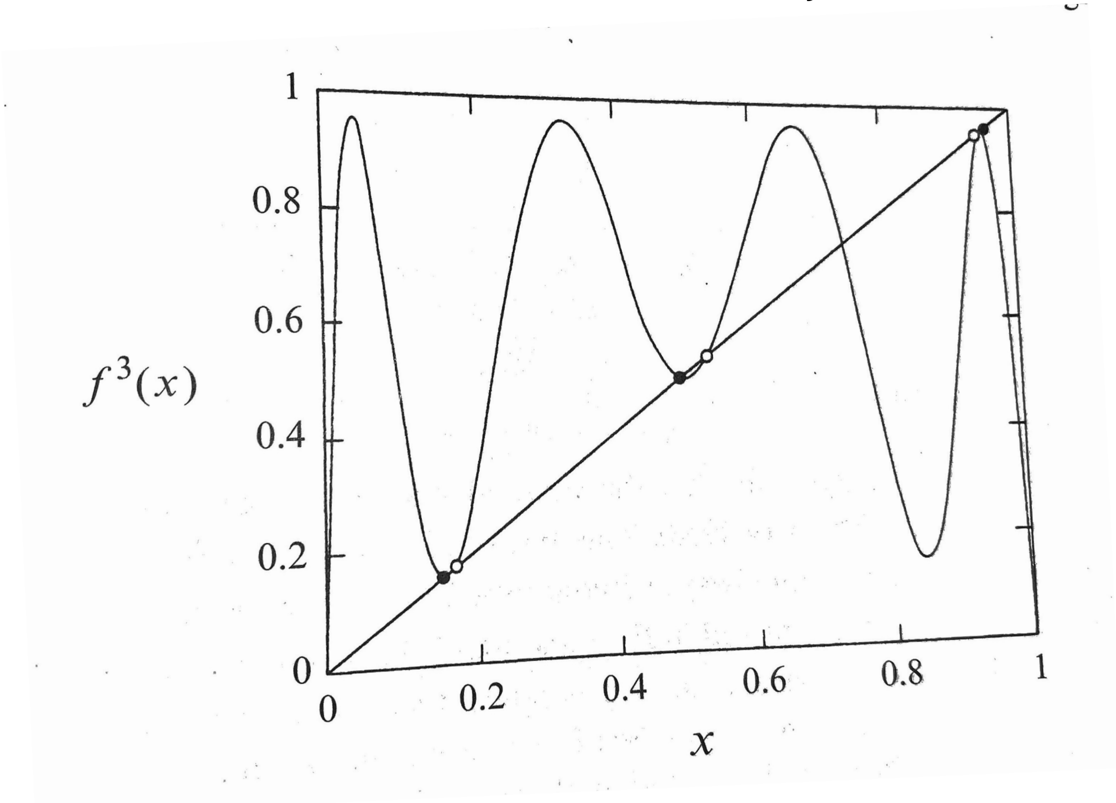
There is a period-3 window for $3.8284 \ldots \leq r \leq 3.8415 \ldots$

Title of this slide is the title of a famous paper by Tien-Yien Li and James York.



How Does This Period-3 Window Emerge?

It we write the logistic equation as $x_{n+1} = f(x_n)$, then the period-3 orbit points are fixed points of the third-iterate map $f^3(x)$.

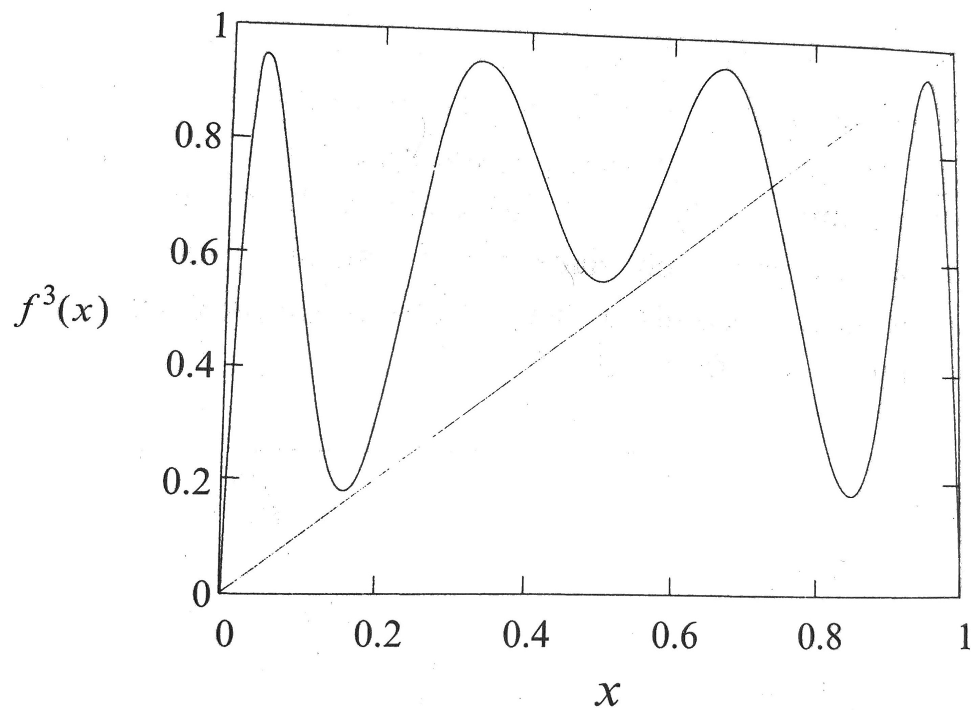


$r = 3.835$

There are 8 real roots: 2 correspond to unstable fixed points of the logistic equation, 3 correspond to the stable 3-cycle (filled circles), and 3 to an unstable 3-cycle (open circles).

How Does This Period-3 Window Emerge?

Now reduce r slightly; the hills move down and the valleys move up. The neighboring fixed points of the third-iterate map coalesce at a tangent bifurcation (similar to a saddle-node of periodics bifurcation). For an even smaller value of r we have:

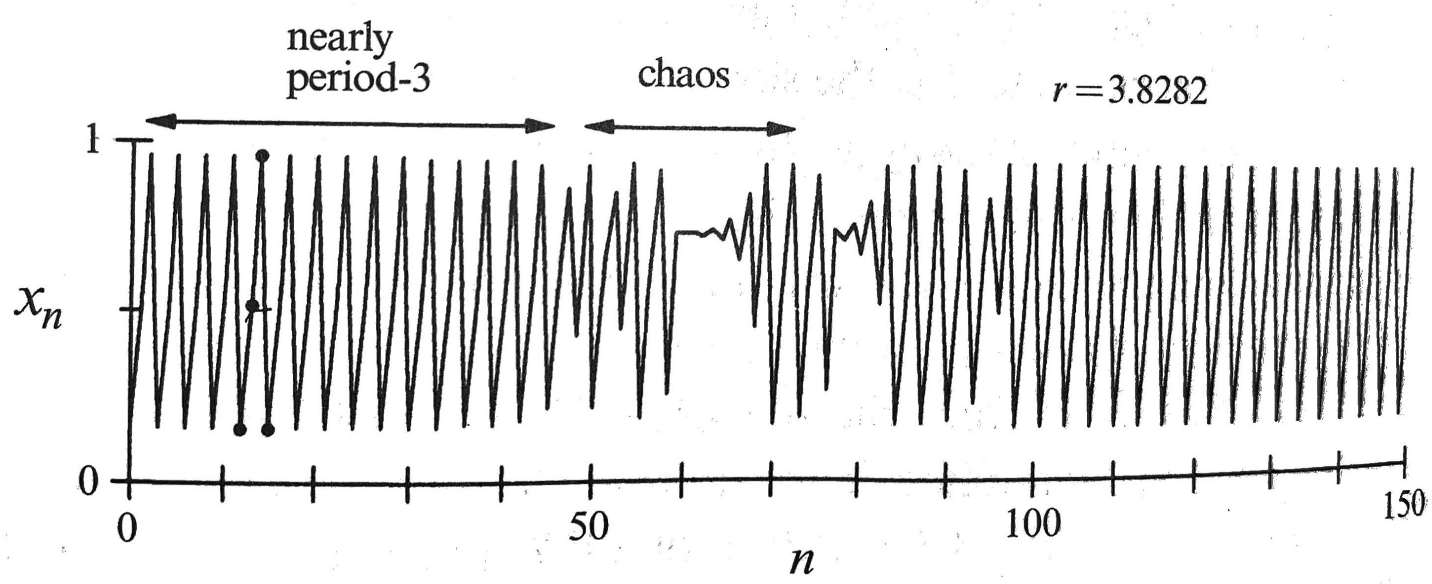


$r = 3.8$

There are 2 real roots, both correspond to unstable fixed points of the logistic equation. The stable and unstable 3-cycles have coalesced and disappeared.

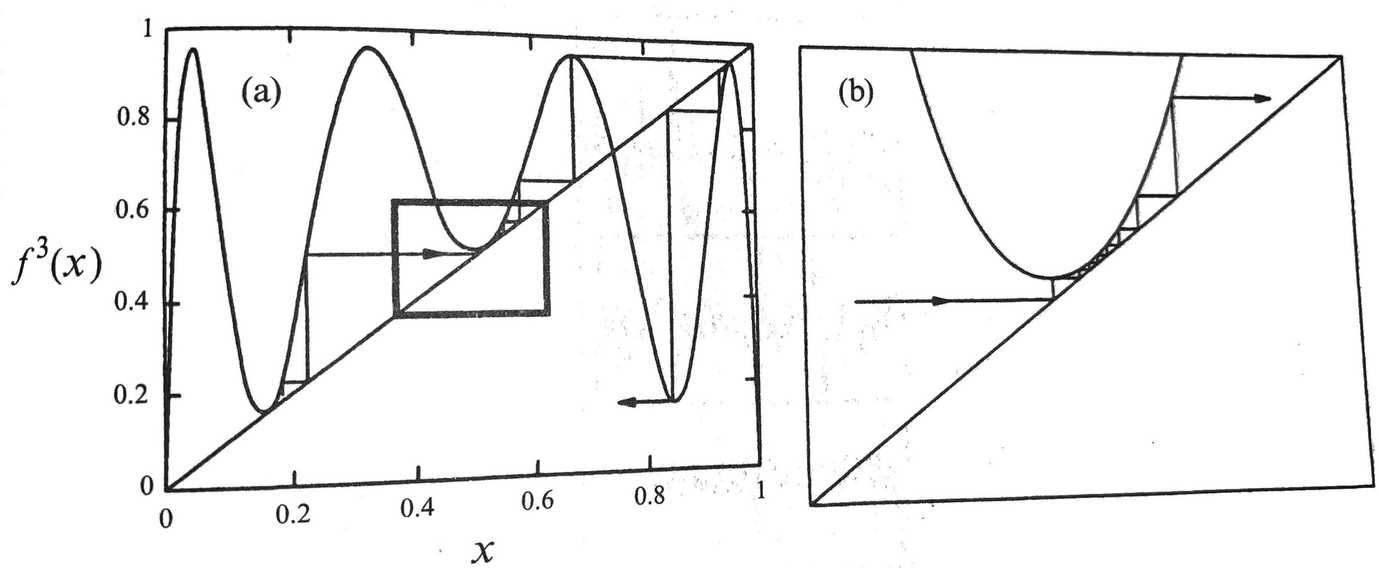
Intermittency Near the Tangent Bifurcation Point

Intermittency occurs for r just a bit smaller than the tangent bifurcation point.



The time course looks mostly like a period-3 oscillation, but short chaotic intervals occur at seemingly random times and for random durations.

The intermittency is Due to Ghosts!



There are three narrow channels near where the 3-cycles will appear if r is increased a bit more. These are ghosts of the 3-cycles. When the trajectory enters any one of these channels a cycle of almost period 3 occurs, until the trajectory leaves the channel. After this, the orbit becomes chaotic, until another channel is entered.

Intermittency Route to Chaos

Intermittency is a generic behavior of systems that become chaotic at saddle-node of periodics-like bifurcations. For example, it is also seen in the Lorenz equations. The progression from periodic, to intermittent chaos, to full chaos is called the intermittency route to chaos.

Recall that the period-3 window is $3.8284\ldots \leq r \leq 3.8415\ldots$ Intermittency occurs for r near (but below) 3.8284. What happens for r near (but above) 3.8415? Here you get a period-doubling cascade, yielding oscillations with period 3×2^p, and beyond the period-doubling limit there is chaos again. Thus, at the right boundary of the period-3 interval there is a period-doubling route to chaos!

The End