Chapter 3 – Voting and Elections

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Sec 3.1 – Voting Systems

Goals
- Study voting systems
  - Plurality method
  - Borda count method
  - Plurality with elimination method
  - Pairwise comparison method
- Discuss tie-breaking methods

Plurality Method

When a candidate receives more than half of the votes in an election, we say the candidate has received a ____________ of the votes.

When a candidate receives the greatest number of votes in an election, but not more than half, we say the candidate has received a ____________ of the votes.

In the plurality method:
- Voters vote for ____________ candidate.
- The candidate receiving the ________________ wins.

This method has a couple advantages:
- The voter chooses only _____ candidate.
- The winner is ____________________________.

The plurality method is used:
- In the United States to elect senators, representatives, governors, judges, and mayors.
- In the United Kingdom and Canada to elect members of parliament.
Borda Count Method

In the Borda count method:

- Voters rank all of the $m$ candidates.
- Votes are counted as follows:
  - A voter’s last choice gets _____ point.
  - A voter’s next-to-last choice gets _____ points.
  - ...
  - A voter’s first choice gets _____ points.
- The candidate with the _______________________ wins.

The main advantage of the Borda count method is that it uses _______________________ from the voters.

A variation of the Borda count method is used to select the winner of the Heisman trophy.

Use Homework #4 (pg 154) preference table. Determine the winner by Borda Count method.
Plurality with Elimination Method

In the plurality with elimination method:

• Voters choose ________ candidate.

• The votes are counted.

  • If one candidate receives a __________ of the votes, that candidate is selected.

  • If no candidate receives a majority, eliminate the candidate who received the __________ votes and do another round of voting.

  • This process is repeated until someone receives a _________ of the votes and is declared the winner.

The plurality with elimination method is used:

• To select the location of the Olympic games.

• In France to elect the president.

To avoid multiple votes:

• Rather than needing to potentially conduct multiple votes, the voters can be asked to rank all candidates during the first election.

• A preference table is used to display these rankings.

**Example 1:** Find the plurality with elimination winner for this election:

<table>
<thead>
<tr>
<th># of votes</th>
<th>33</th>
<th>39</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>A</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
<td>B</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>C</td>
<td>C</td>
<td>B</td>
</tr>
</tbody>
</table>
Example 2:

The table below shows the results when the members of a committee were asked to vote on a kind of pet to donate to a local preschool. The choices were alligator (A), barnacle (B), cobra (C), and dingo (D). Each voter ranked his or her choices from first to fourth. The top row of the table shows the number of votes for the ranking order in each column. Under the plurality with elimination method, who is the winner?

<table>
<thead>
<tr>
<th>Ranking</th>
<th>10</th>
<th>3</th>
<th>9</th>
<th>6</th>
<th>1</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>C</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>2nd</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td>A</td>
<td>D</td>
<td>C</td>
</tr>
<tr>
<td>3rd</td>
<td>D</td>
<td>C</td>
<td>A</td>
<td>D</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>4th</td>
<td>B</td>
<td>D</td>
<td>C</td>
<td>B</td>
<td>B</td>
<td>A</td>
</tr>
</tbody>
</table>

Pairwise Comparison Method

In the pairwise comparison method:

- Voters rank all of the candidates.

- For each pair of candidates $X$ and $Y$, determine how many voters prefer $X$ to $Y$ and vice versa.
  - If $X$ is preferred to $Y$ more often, $X$ gets 1 point.
  - If $Y$ is preferred to $X$ more often, $Y$ gets 1 point.
  - If the candidates tie, each gets $\frac{1}{2}$ a point.

- The candidate with the most points wins.

- The pairwise comparison method is also called the Condorcet method.
Example:

The table below shows the results when the members of a committee were asked to vote on a mode of transportation to carry a group to a conference on energy conservation. The choices were armored car (A), blimp (B), clipper ship (C), and dump truck (D). Each voter ranked his or her choices from first to fourth. The top row of the table shows the number of votes for the ranking order in each column. Find the winner according to the Pairwise comparison method.

<table>
<thead>
<tr>
<th>Ranking</th>
<th>7</th>
<th>2</th>
<th>4</th>
<th>1</th>
<th>9</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>D</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td>A</td>
</tr>
<tr>
<td>3rd</td>
<td>D</td>
<td>B</td>
<td>C</td>
<td>D</td>
<td>A</td>
<td>C</td>
</tr>
<tr>
<td>4th</td>
<td>C</td>
<td>C</td>
<td>D</td>
<td>A</td>
<td>D</td>
<td>B</td>
</tr>
</tbody>
</table>
The four voting systems studied here can produce different winners even when the same voter preference table is used.

Any of the four methods can also produce a tie between two or more candidates, which must be broken somehow.

**Tie Breaking**

- A tie-breaking method should be chosen _________ the election.

- To break a tie caused by perfectly balanced voter support, election officials may:
  - Make an ______________ choice.
    - Flipping a coin
    - Drawing straws
  - Bring in ____________________.
    - The ______________ votes when the U. S. Senate is tied.

There are many situations in which there is a need to know not only the winner of the election, but also how the other candidates ranked.

Each of the methods we have studying can be extended to produce a winner and rank the other candidates for second, third place, etc.

**Sec 3.2 – Flaws of the Voting Systems**

**Goals**

- Study fairness criteria
  - The majority criterion
  - Head-to-head criterion
  - Montonicity criterion
  - Irrelevant alternatives criterion

- Study fairness of voting methods
  - Arrow impossibility theorem
  - Approval voting

We have seen that the choice of voting method can affect the outcome of an election.

Each voting method studied can fail to satisfy certain criteria that make a voting method “fair”.

6
The ________________ are properties that we expect a good voting system to satisfy.

Four fairness criteria will be studied:
• The majority criterion
• The head-to-head criterion
• The monotonicity criterion
• The irrelevant alternatives criterion

The Majority Criterion

If a candidate is the first choice of a ___________________________, then that candidate should be selected.

➢ If candidate X has a majority but candidate X is not the winner, then the majority criterion has been violated.

➢ If no one has a majority, this criterion is not violated.

➢ If candidate X has a majority and candidate X is the winner, then the majority criterion is not violated.

• If a candidate is the first choice of a majority of voters, then that candidate ________________ win using:
  o The plurality method.
  o The plurality with elimination method.
  o The pairwise comparison method.
    ♣ In all of these methods any candidate with more than half the vote will always win.

• If a candidate is the first choice of a majority of voters, then that candidate ________________ win using:
  o The Borda count method.
    ♣ The candidate with the most points may not be the candidate with the most first-place votes.

Work Homework #2 (pg 172)
1) Determine the winner by Borda Count method.
2) Has the Majority Criterion been violated?
The Head-to-Head Criterion

- If a candidate is favored when compared separately with each of the other candidates, then the favored candidate should win the election.
  - This is also called the Condorcet criterion.

- If candidate X wins all head to head comparisons but candidate X is not the winner, then the head to head criterion is violated.

- If no candidate is favored in all head to head comparisons, this criterion is not violated.

- If candidate X wins all head to head comparisons and candidate X is the winner, then the head to head criterion is not violated.

- If a candidate is favored pairwise to every other candidate, then that candidate win using:
  - The pairwise comparison method.
    - This candidate will earn the most points from the pairwise comparisons.

- If a candidate is favored pairwise to every other candidate, then that candidate might using:
  - The plurality method.
  - The plurality with elimination method.
  - The Borda count method.

Work Homework #4 (pg 173)

1) Determine the winner by Plurality method.
2) Has the Head to Head criterion been violated?
The Monotonicity Criterion

• Suppose a particular candidate, X, wins an election.

• If, hypothetically, this election were redone and the only changes were that some voters switched X with the candidate they had ranked one higher, then X should still win.
  o This criterion is only used in special cases.

• The monotonicity criterion is always satisfied by:
  o The plurality method.
  o The Borda count method.
  o The pairwise comparison method.

• The monotonicity criterion is not always satisfied by:
  o The plurality with elimination method.

Work Homework #20 (pg 176)

The Irrelevant Alternatives Criterion

• Suppose a candidate, X, is selected in an election.

• If, hypothetically, this election were redone with one or more of the unselected candidates removed from the vote, then X should still win.

• The irrelevant alternatives criterion is not always satisfied by any of the 4 voting methods studied.

Work Homework #12 (pg 174)
**Question 1:**

The table below shows the results when the members of an organization were asked to select an editor from a slate of three candidates. Each voter ranked his or her choices from first to third. The top row of the table shows the number of votes for the ranking order in each column. The winner was determined by Borda count. The winner is Berta. Select 'Yes' if this election violates the Majority criterion. Select 'No' if this election doesn’t violate the Majority criterion.

<table>
<thead>
<tr>
<th>Ranking</th>
<th>10</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>Carla</td>
<td>Carla</td>
<td>Berta</td>
<td>Berta</td>
<td>Nell</td>
<td>Nell</td>
</tr>
<tr>
<td>2nd</td>
<td>Berta</td>
<td>Nell</td>
<td>Carla</td>
<td>Nell</td>
<td>Carla</td>
<td>Berta</td>
</tr>
<tr>
<td>3rd</td>
<td>Nell</td>
<td>Berta</td>
<td>Nell</td>
<td>Carla</td>
<td>Berta</td>
<td>Carla</td>
</tr>
</tbody>
</table>

(a) Yes  
(b) No

**Question 2:**

The table below shows the results when the members of a committee were asked to vote on a mode of transportation to carry a group to a conference on energy conservation. The choices were armored car (A), blimp (B), choo choo train (C), and donkey cart (D). Each voter ranked his or her choices from first to fourth. The top row of the table shows the number of votes for the ranking orders in each column. The winner, chosen by Borda count, was armored car. Select 'Yes' if this election violates the head-to-head criterion, 'No' if the election doesn’t violate the head-to-head criterion.

<table>
<thead>
<tr>
<th>Ranking</th>
<th>3</th>
<th>11</th>
<th>5</th>
<th>10</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>C</td>
<td>C</td>
<td>A</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>3rd</td>
<td>D</td>
<td>B</td>
<td>D</td>
<td>B</td>
<td>D</td>
<td>A</td>
</tr>
<tr>
<td>4th</td>
<td>C</td>
<td>D</td>
<td>B</td>
<td>D</td>
<td>B</td>
<td>D</td>
</tr>
</tbody>
</table>

(a) Yes  
(b) No
Question 3:

The table below shows the results when the members of an organization were asked to select a telephone sanitizer from a slate of three candidates. Each voter ranked his or her choices from first to third. The top row of the table shows the number of votes for the ranking order in each column. The winner was determined by the pairwise comparison method. The winner is Berta. Select 'Yes' if this election violates the Irrelevant Alternatives criterion. Select 'No' if this election doesn’t violate the Irrelevant Alternatives criterion.

<table>
<thead>
<tr>
<th>Ranking</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>Vicki</td>
<td>Vicki</td>
<td>Liz</td>
<td>Liz</td>
<td>Berta</td>
<td>Berta</td>
</tr>
<tr>
<td>2nd</td>
<td>Liz</td>
<td>Berta</td>
<td>Vicki</td>
<td>Berta</td>
<td>Vicki</td>
<td>Liz</td>
</tr>
<tr>
<td>3rd</td>
<td>Berta</td>
<td>Liz</td>
<td>Berta</td>
<td>Vicki</td>
<td>Liz</td>
<td>Vicki</td>
</tr>
</tbody>
</table>

(a) Yes
(b) No

Question 4:

A flock of butterflies voted on their favorite color of flower. The table above shows the results. Each voter ranked his or her top two choices. The winner under the plurality with elimination is orange. Because of complaints that the butterfly ballot was confusing, a Federal judge ordered a do-over. In the new election, everybody voted the same way they had voted in the first election, except that 7 voters switched their first preference from red to orange. The preference table for the second election is shown below. Select 'Yes' if this situation violates the Monotonicity Criterion. Select 'No’ otherwise.

<table>
<thead>
<tr>
<th></th>
<th>26</th>
<th>29</th>
<th>42</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>red</td>
<td>yellow</td>
<td>orange</td>
</tr>
<tr>
<td>2nd</td>
<td>yellow</td>
<td>orange</td>
<td>red</td>
</tr>
</tbody>
</table>

(a) Yes
(b) No
Study Table 3.25 (pg 170)

Arrow Impossibility Theorem

- The Arrow Impossibility Theorem states that no system of voting will always satisfy all of the 4 fairness criteria.
  - This fact was proved by Kenneth Arrow in 1951.

Approval Voting

No voting system is ____________________, but we can explore systems that are unfair less often than others. One such system is called approval voting.

In approval voting:
- Each voter votes for ________ candidates he/she considers acceptable.
- The candidate with the most votes is selected.

EXAMPLE:

The table below shows the results when the ten members of the banquet planning committee each submitted an approval ballot for a choice of entree. The choices were lasagna, baked chicken, roast beef, fried fish, and curried red ants. For each option, each voter marked 'X' for approval, and left the ballot unmarked for disapproval. Find the winner.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>lasagna</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
</tr>
<tr>
<td>baked chicken</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>roast beef</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fried fish</td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>curried red ants</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td></td>
</tr>
</tbody>
</table>
Sec 3.3 – Weighted Voting Systems

Goals
• Study weighted voting systems
  o Coalitions
  o Dummies and dictators
  o Veto power
• Study the Banzhaf power index

Weighted Voting System
• In a weighted voting system, an individual voter may have __________________ vote.

• The number of votes that a voter controls is called the ______ of the voter.
  o An example of a weighted voting system is the election of the U.S. President by the Electoral College.

• The weights of the voters are usually listed as a sequence of numbers between square brackets.

• For example, the voting system in which Angie has a weight of 9, Roberta has a weight of 12, Carlos has a weight of 8, and Darrell has a weight of 11 is represented as [12, 11, 9, 8].

• The voter with the ______ weight is called the “first voter”, written P₁.

• The weight of the first voter is represented by W₁.

• The remaining voters and their weights are represented similarly, in order of decreasing weights.

• Yes or no questions are commonly called ____________.

• A final decision of ‘No’ ___________ the motion and leaves the status quo unchanged.

• A final decision of ‘Yes” __________ the motion and changes the status quo.

• A __________________________ requirement means that a motion must receive more than half of the votes to pass.

• A __________________________ requirement means that the minimum number of votes required to pass a motion is set higher than half of the total weight.
  o A common super majority is two-thirds of the total weight.
o The weight required to pass a motion is called the ________.

o Example: A simple majority quota for the weighted voting system [12, 11, 9, 8] would be 21.

o Half of the total weight is $(12 + 11 + 9 + 8)/2 = 40/2 = 20$. More than half of the weight would be at least 21 'Yes' votes.

o The quota for a weighted voting system is usually included in the list of weights.

o Example: For the weighted voting system [12, 11, 9, 8] with a quota of 21 the complete notation is [21 | 12, 11, 9, 8].

Work Homework #2, #4c, #6c, #8bc (pg 193)
Coalitions

- Any nonempty subset of the voters in a weighted voting system is called a ____________.

- If the total weight of the voters in a coalition is greater than or equal to the quota, it is called a _________ coalition.

- If the total weight of the voters in a coalition is less than the quota, it is called a _________ coalition.

- In a weighted voting system with $n$ voters, exactly $2^n - 1$ coalitions are possible.

Example:
- How many coalitions are possible in a weighted voting system with 7 voters?

Solution: The formula tells us there are __________________________ coalitions.

Work Homework #12bc, #16a, #18a (pg 193, 194)
#18a

<table>
<thead>
<tr>
<th>Coalition</th>
<th>Total weight</th>
<th>Winning or Losing?</th>
<th>Critical voters?</th>
</tr>
</thead>
<tbody>
<tr>
<td>{P_1}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{P_2}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{P_3}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{P_1, P_2}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{P_1, P_3}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{P_2, P_3}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{P_1, P_2, P_3}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Critical Voters**

- If a voter’s weight is large enough so that the voter can change a particular winning coalition to a losing coalition by leaving the coalition, then that voter is called a critical voter in that winning coalition.

Let’s go back to #18 (above) and determine the critical voters for each winning coalition.

**The Banzhaf Power Index**

- The more times a voter is a ________________ in a system, the more power that voter has in the system.

- The ________________ power of a voter is the number of winning coalitions in which that voter is critical.

- The sum of the Banzhaf powers of all voters is called the total Banzhaf power in the weighted voting system.

- An individual voter’s Banzhaf power index is the _____ of the voter’s Banzhaf power to the total Banzhaf power in the system.  
  - The sum of the Banzhaf power indices of all voters is ___

- An individual voter’s Banzhaf power index is calculated using the following process:

  1. Find all winning coalitions for the system.
2. Determine the critical voters for each winning coalition.
3. Calculate each voter’s Banzhaf power.
4. Find the total Banzhaf power in the system.
5. Divide each voter’s Banzhaf power by the total Banzhaf power.

Compare these weighted voting systems by looking at their critical votes & Banzhaf Power Indices:

LOOK back at #18a and determine the Banzhaf power index for each voter. Did \( P_2 \) have more power than \( P_3 \)?

\[
\begin{align*}
#1 & \ [5 \mid 5, 3, 1] \\
#2 & \ [5 \mid 5, 5, 3]
\end{align*}
\]

Dictators and Dummies

- A voter whose presence or absence in any coalition makes no difference in the outcome is called a dummy. A dummy has Banzhaf Power Index ________.

- A voter whose presence or absence in any coalition completely determines the outcome is called a dictator. A dictator has Banzhaf Power Index ________.

- When a weighted voting system has a dictator, the other voters in the system are automatically dummies.
Veto Power

- In between the complete power of a dictator and the zero power of a dummy is a level of power called veto power.

- A voter with veto power can defeat a motion by voting ‘No’ but cannot necessarily pass a motion by voting ‘Yes’.

- Any dictator has veto power, but a voter with veto power is not necessarily a dictator.

- A voter with veto power is critical to every winning coalition.

Let’s go back to the 3 systems we compared and list those with veto power, dummies, and dictators.
For any 4 voter system, these are the coalitions you need to consider:

<table>
<thead>
<tr>
<th>Coalition</th>
</tr>
</thead>
<tbody>
<tr>
<td>{P_1}</td>
</tr>
<tr>
<td>{P_2}</td>
</tr>
<tr>
<td>{P_3}</td>
</tr>
<tr>
<td>{P_4}</td>
</tr>
<tr>
<td>{P_1, P_2}</td>
</tr>
<tr>
<td>{P_1, P_3}</td>
</tr>
<tr>
<td>{P_1, P_4}</td>
</tr>
<tr>
<td>{P_2, P_3}</td>
</tr>
<tr>
<td>{P_2, P_4}</td>
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<tr>
<td>{P_3, P_4}</td>
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<td>{P_1, P_2, P_3}</td>
</tr>
<tr>
<td>{P_1, P_2, P_4}</td>
</tr>
<tr>
<td>{P_1, P_3, P_4}</td>
</tr>
<tr>
<td>{P_2, P_3, P_4}</td>
</tr>
<tr>
<td>{P_1, P_2, P_3, P_4}</td>
</tr>
</tbody>
</table>

Consider the weighted voting system given by

\[[13 \mid 7,5,3,3]\]

Calculate the Banzhaf index for \(P_1\)

(a) 0.400
(b) 0.000
(c) 0.600
(d) 0.133