1. Let $X$ and $Y$ be sets. Prove the following statements.

(i) If there is a surjection $f: X \to Y$, then $|X| \geq |Y|$.

(ii) If there exists surjections $f: X \to Y$ and $g: Y \to X$, then $|X| = |Y|$.

(iii) If there exists an injection $f: X \to Y$ and a surjection $g: X \to Y$, then $|X| = |Y|$.

2. Let $X$ be a set. The **power set** of $X$ is denoted as $2^X$ and is defined as the set of all subsets of $X$. Thus, $A \subset X$ means exactly the same as $A \in 2^X$. For example, if $X = \{1, 2, 3\}$,

\[
2^X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.
\]

(i) If $X$ is finite with $|X| = n \in \mathbb{N} \cup \{0\}$, prove that $|2^X| = 2^n$.

(ii) For any sets $X$ and $Y$, prove that $|X| = |Y| \Rightarrow |2^X| = |2^Y|$.

3. Prove that for any set $X$, $|X| < |2^X|$.

   Hint: For any function $g: X \to 2^X$, define $A_g = \{x \in X : x \notin g(x)\}$. Show that $A_g \notin g(X)$, hence $g$ is not surjective.

4. Let $f: X \to Y$ and $g: Y \to Z$ be functions. Prove the following statements.

   (i) $f$ and $g$ injective $\Rightarrow g \circ f$ injective.

   (ii) $f$ and $g$ surjective $\Rightarrow g \circ f$ surjective.

   (iii) $g \circ f$ injective $\Rightarrow f$ injective.

   (iv) $g \circ f$ surjective $\Rightarrow g$ surjective.

   (v) Give specific examples to show that $g$ need not be injective whenever $g \circ f$ is injective and that $f$ need not be surjective whenever $g \circ f$ is surjective.

   (vi) For any subset $C \subset Z$, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

   (vi) For $A \subset X$ and $B \subset Y$, $A \subset f^{-1}(f(A))$ and $f(f^{-1}(B)) \subset B$. 
5. A **binary sequence** is an infinite list of zeroes and ones, i.e., \( b \) is a binary sequence means that \( b = b_1, b_2, b_3, \ldots, b_i, \ldots \), where each \( b_i \in \{0, 1\} \). Let \( B \) be the set of all binary sequences. Let \( A \) be the subset of \( B \) containing only those sequences that have finitely many ones and let \( C \) be the subset of \( B \) containing only those sequences that have no consecutive ones. Thus,

\[
A = \{ b \in B : \exists N \in \mathbb{N} \text{ such that } b_i = 0 \ \forall i \geq N \}, \\
C = \{ b \in B : \forall i, b_i = 1 \implies b_{i+1} = 0 \}.
\]

(i) Use a Cantor diagonal argument to show that \( B \) is uncountable.

(ii) Prove that \( A \) is countably infinite.

(iii) Decide whether or not \( C \) is countable and prove your contention.