MAA 4226 Advanced Calculus | **Homework Set 1** | **Bowers Fall 2000**

1. Prove that $\sqrt{6}$ is irrational.

The Field Axioms:

Let $\mathbf{F}$ be a set with two binary operations $+$ and $\times$ that satisfy:

FA0: (closure) For all a, b in $\mathbf{F}$, $a + b$ and $a \times b$ are in $\mathbf{F}$.

FA1: (commutative) For all a, b in $\mathbf{F}$, $a + b = b + a$ and $a \times b = b \times a$.

FA2: (associative) For all a, b, c in $\mathbf{F}$, $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$.

FA3: (identities) An additive identity 0 and a multiplicative identity 1 are in $\mathbf{F}$: for all a in $\mathbf{F}$, $a + 0 = a$ and $a \times 1 = a$. To avoid trivialities, we require that $0 \neq 1$.

FA4: (inverses) For all a and nonzero b in $\mathbf{F}$, there exists in $\mathbf{F}$ an additive inverse u for a and a multiplicative inverse v for b: $a + u = 0$ and $b \times v = 1$. Notation: u is denoted $-a$ and v is denoted b^{-1} or $\frac{1}{b}$.

FA5: (distributive) For all a, b, c in $\mathbf{F}$, $a \times (b + c) = a \times b + a \times c$.

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2. Let $\mathbf{F} = \{x + y\sqrt{7} : x, y \in \mathbf{Q}\}$ with the usual addition and multiplication operations. Since $\mathbf{F}$ is a subset of the real numbers $\mathbf{R}$, FA1, FA2, and FA5 are automatically satisfied. Show that the remaining field axioms are satisfied so that $\mathbf{F}$ is a field.

3. Let $\mathbf{F} = (0, \infty) = \{x \in \mathbf{R} : x > 0\}$, the set of positive real numbers. We use the symbols $\oplus$ and $\otimes$ to denote the following operations on $\mathbf{F}$: for $a, b \in \mathbf{F}$, $a \oplus b = ab$ and $a \otimes b = a^{\ln b}$. For items (i) and (ii) below, you might find useful the calculus formulas $a^x = e^{x \ln a}$, $e^{\ln x} = x$, $\ln e^x = x$, and the laws of exponents and logarithms.

   (i) Show that $\mathbf{F}$ is a field under $\oplus$-addition and $\otimes$-multiplication.

   (ii) Define $h: \mathbf{R} \to \mathbf{F}$ by $h(x) = e^x$. Show that for any two reals $a, b \in \mathbf{R}$,

$$h(a + b) = h(a) \oplus h(b) \qquad \text{and} \qquad h(ab) = h(a) \otimes h(b).$$

Definitions. A set $\mathbf{F}$ is a *ring* if addition and multiplication operations are defined on $\mathbf{F}$ that satisfy all the field axioms except that multiplicative inverses are not required to exist for every element of $\mathbf{F}$. The standard example is the ring of integers $\mathbf{Z}$. A nonzero element a of $\mathbf{F}$ is a *zero divisor* if there is a nonzero element b for which $ab = 0$. The ring of integers has no zero divisors.

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4. Let $\mathbf{F}$ be a ring.

   (i) Show that if a is a zero divisor of $\mathbf{F}$, then a fails to have a multiplicative inverse.

   (ii) Show that a field has no zero divisors.

5. Construct Cayley tables for addition and multiplication in $\mathbf{Z}/4$, $\mathbf{Z}/5$, and $\mathbf{Z}/8$.

6. How, by just looking at a Cayley table for multiplication, does one determine if an element has a multiplicative inverse? Which of the rings of modular integers from Problem 5 are fields?

7. For what values of n does $\mathbf{Z}/n$ have zero divisors? Why? Explain fully.

8. We have already seen that if a is a zero divisor in a ring $\mathbf{D}$, then a does not have a multiplicative inverse (therefore, $\mathbf{D}$ is not a field). The converse of this fact is the statement

   If the element a of the ring $\mathbf{D}$ does not have a multiplicative inverse, then a must be a zero divisor.

   (i) Give an example of a ring where this converse statement is false. HINT: Look at the most natural ring around!

   (ii) Prove that if $\mathbf{D}$ is a finite ring, ie, $\mathbf{D}$ has only finitely many elements, then this converse statement is true.

   (iii) From Problem 7 and item (ii) above, for which n is $\mathbf{Z}/n$ a field?

9. Find a field with four elements. HINT: Call such a field $\mathbf{F}$ and let 0 and 1 denote the additive and multiplicative inverses. Call the remaining elements of $\mathbf{F}$ a and b so that $\mathbf{F} = \{0, 1, a, b\}$. Use the field axioms to fill in as much of the Cayley tables for addition and multiplication as you can and work with equations to see if the whole table is forced on you.

10. Write each of the following elements of $\mathbf{Z}/13$ as $\overline{a}$ for some $a \in \{0, 1, \ldots, 12\}$.

    (i) $-\overline{7}$ (ii) $\overline{10}^{-1}$ (iii) $\overline{4}^2 - \overline{6}$ (iv) $(\overline{131})^{10}$ (v) $-\overline{14}$

11. Which elements of $\mathbf{Z}/7$ have 4th roots? Find all 4th roots of elements that have them.

12. Find all the square roots of $\overline{4}$ in $\mathbf{Z}/5$, then in $\mathbf{Z}/7$.

13. Find the sum of all the elements of $\mathbf{Z}/n$, ie, find $\overline{0} + \overline{1} + \cdots + \overline{n-1}$. Now find the product of all the nonzero elements of $\mathbf{Z}/n$, ie, find $\overline{1} \times \overline{2} \times \cdots \times \overline{(n-1)} = \overline{(n-1)!}$.

14. Solve the equation $x^2 + \overline{2}x - \overline{3} = \overline{0}$ in $\mathbf{Z}/5$. Does it make sense to apply the quadratic formula? Now solve the equation $\overline{2}x^2 + \overline{1} = \overline{0}$ in $\mathbf{Z}/5$. What happened?