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1. Let X and Y be sets. Prove the following statements.

   (i) If there is a surjection $f: X \to Y$, then $|X| \geq |Y|$.

   (ii) If there exists surjections $f: X \to Y$ and $g: Y \to X$, then $|X| = |Y|$.

   (iii) If there exists an injection $f: X \to Y$ and a surjection $g: X \to Y$, then $|X| = |Y|$.

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2. Let X be a set. The **power set** of X is denoted as 2^X and is defined as the set of all subsets of X. Thus, $A \subset X$ means exactly the same as $A \in 2^X$. For example, if $X = \{1, 2, 3\}$, then

$$2^X = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

   (i) If X is finite with $|X| = n \in \mathbf{N} \cup \{0\}$, prove that $|2^X| = 2^n$.

   (ii) For any sets X and Y, prove that $|X| = |Y| \Rightarrow |2^X| = |2^Y|$.

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3. Prove that for any set X, $|X| < |2^X|$.

   HINT: For any function $g: X \to 2^X$, define $A_g = \{x \in X : x \notin g(x)\}$. Show that $A_g \notin g(X)$, hence g is not surjective.

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4. Let $f: X \to Y$ and $g: Y \to Z$ be functions. Prove the following statements.

   (i) f and g injective $\Rightarrow g \circ f$ injective.

   (ii) f and g surjective $\Rightarrow g \circ f$ surjective.

   (iii) $g \circ f$ injective $\Rightarrow f$ injective.

   (iv) $g \circ f$ surjective $\Rightarrow g$ surjective.

   (v) Give specific examples to show that g need not be injective whenever $g \circ f$ is injective and that f need not be surjective whenever $g \circ f$ is surjective.

   (vi) For any subset $C \subset Z$, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

   (vi) For $A \subset X$ and $B \subset Y$, $A \subset f^{-1}(f(A))$ and $f(f^{-1}(B)) \subset B$.

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5. A **binary sequence** is an infinite list of zeroes and ones, ie, b is a binary sequence means that $b = b_1, b_2, b_3, \ldots, b_i, \ldots$, where each $b_i \in \{0, 1\}$. Let B be the set of all binary sequences. Let A be the subset of B containing only those sequences that have finitely many ones and let C be the subset of B containing only those sequences that have no consecutive ones. Thus,

$$A = \{b \in B \colon \exists N \in \mathbf{N} \text{ such that } b_i = 0 \ \forall i \geq N\},$$
$$C = \{b \in B \colon \forall i, b_i = 1 \Rightarrow b_{i+1} = 0\}.$$

(i) Use a Cantor diagonal argument to show that B is uncountable.

(ii) Prove that A is countably infinite.

(iii) Decide whether or not C is countable and prove your contention.
