A function $f: X \to Y$ induces a function $f: 2^X \to 2^Y$ defined by: for $A \in 2^X, f(A) = \{f(a): a \in A\} \in 2^Y$. Note that we use the same notation for these two functions though technically we should use some other notation for the induced function. Usually this causes no confusion as the context makes clear the meaning. There is another induced function $f^{-1}: 2^Y \to 2^X$ defined by: for $B \in 2^Y, f^{-1}(B) = \{x \in X: f(x) \in B\} \in 2^X$. For $A \in 2^X$, $f(A)$ is called the **image of** A **under** f and for $B \in 2^Y$, $f^{-1}(B)$ is called the **pre-image of** B **under** f. For a singleton set $\{y\}$, we usually write $f^{-1}(y)$ instead of $f^{-1}(\{y\})$, and $f^{-1}(y)$ is called the **fiber of** f **over** y.

Let $\mathcal{C} \subset 2^X$ and $\mathcal{D} \subset 2^Y$. Thus, $\mathcal{C}$ is a collection of subsets of X, ie, each $C \in \mathcal{C}$ satisfies $C \subset X$, and similarly for $\mathcal{D}$. We make the following definitions, where $A, B \in 2^X$.

$$\begin{aligned} f(\mathcal{C}) &= \{f(C): C \in \mathcal{C}\} \subset 2^Y \\ f^{-1}(\mathcal{D}) &= \{f^{-1}(D): D \in \mathcal{D}\} \subset 2^X \\ \cup\mathcal{C} &= \{x \in X: \exists C \in \mathcal{C} \text{ such that } x \in C\} \\ \cap\mathcal{C} &= \{x \in X: \forall C \in \mathcal{C}, x \in C\} \\ A - B &= \{x \in X: x \in A \text{ and } x \notin B\} \end{aligned}$$

$\cup\mathcal{C}$ is the **union** of $\mathcal{C}$, $\cap\mathcal{C}$ is the **intersection** of $\mathcal{C}$, and $A - B$ is the **set difference** between A and B.

Prove the following statements for $f: X \to Y$; $A, B \in 2^X$; $C, D \in 2^Y$; $\mathcal{C} \subset 2^X$; $\mathcal{D} \subset 2^Y$.

1. $f^{-1}(\cup\mathcal{D}) = \cup f^{-1}(\mathcal{D})$.
2. $f^{-1}(\cap\mathcal{D}) = \cap f^{-1}(\mathcal{D})$.
3. $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.
4. $f(\cup\mathcal{C}) = \cup f(\mathcal{C})$.
5. $f(\cap\mathcal{C}) \subset \cap f(\mathcal{C})$.
6. $f(A - B) \supset f(A) - f(B)$.

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7. Give examples to show that the containments in items 5 and 6 cannot in general be replaced by equalities. Then find conditions on the map f that guarantee equality.

8. The **identity function** $\mathrm{id}_X: X \to X$ on the set X is defined by $\mathrm{id}_X(x) = x$ for all $x \in X$. Let $f: X \to Y$ be a function. Prove that f is a bijection if and only if there exists $g: Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.
