1. The identity function \( \text{id}_X: X \to X \) on the set \( X \) is defined by \( \text{id}_X(x) = x \) for all \( x \in X \). Let \( f: X \to Y \) be a function. Prove that \( f \) is a bijection if and only if there exists \( g: Y \to X \) such that \( g \circ f = \text{id}_X \) and \( f \circ g = \text{id}_Y \).

2. Define \( f: \mathbb{N} \to \mathbb{Z}^* = \{ \alpha \in \mathbb{Z}: \alpha \neq 0 \} \) by

\[
f(\alpha) = (-1)^\alpha \left\lfloor \frac{\alpha + 1}{2} \right\rfloor.
\]

Note that \( f(\alpha) = \alpha/2 \) if \( \alpha \) is even and \( f(\alpha) = -(\alpha + 1)/2 \) if \( \alpha \) is odd. Prove that \( f \) is a bijection by exhibiting a function \( g: \mathbb{Z}^* \to \mathbb{N} \) for which \( g \circ f = \text{id}_\mathbb{N} \) and \( f \circ g = \text{id}_{\mathbb{Z}^*} \).

3. This exercise gives an explicit formula for a bijection from \( \mathbb{N} \) to \( \mathbb{Q}^+ \), the set of positive rationals. It is based on the unique factorization of integers into primes. Let \( f: \mathbb{N} \to \mathbb{Z}^* \) be any bijection (for example, the one of the previous exercise). For \( n \in \mathbb{N} \), let \( n = \text{lp}_1^{\alpha_1} \cdots p_s^{\alpha_s} \) where \( p_1, p_2, \ldots, p_s \) are distinct primes, \( \alpha_i \in \mathbb{N} \), and \( s \geq 0 \) is an integer (\( s = 0 \) means that \( n = 1 \)). Define

\[
g(n) = 1p_1^{f(\alpha_1)} \cdots p_s^{f(\alpha_s)} \in \mathbb{Q}^+.
\]

By uniqueness of prime decompositions, \( g(n) \) defines a function \( g: \mathbb{N} \to \mathbb{Q}^+ \). Notice that \( g(1) = 1 \). Prove that \( g \) is a bijection. Hint: Define \( h: \mathbb{Q}^+ \to \mathbb{N} \) by the following process. For \( r \in \mathbb{Q}^+ \), write \( r = a/b \) where \( a, b \in \mathbb{N} \) are relatively prime integers. Write \( a = 1p_1^{\alpha_1} \cdots p_s^{\alpha_s} \), \( b = 1q_1^{\beta_1} \cdots q_t^{\beta_t} \), and note that \( p_i \neq q_j \) for all \( i \) and \( j \). Define

\[
h(r) = 1p_1^{f^{-1}(\alpha_1)} \cdots p_s^{f^{-1}(\alpha_s)}q_1^{f^{-1}(\beta_1)} \cdots q_t^{f^{-1}(\beta_t)}.
\]

Prove that \( h \circ g = \text{id}_\mathbb{N} \) and \( g \circ h = \text{id}_{\mathbb{Q}^+} \).

4. Let \( X, d \) be a metric space. For each \( x \in X \) and nonempty subsets \( A \) and \( B \) of \( X \), define

\[
d(x, A) = \inf\{d(x, a): a \in A\}
\]
\[
d(A, B) = \inf\{d(a, b): a \in A, b \in B\}.
\]

(i) Prove that \( d(x, A) = 0 \iff x \in \overline{A} \).

(ii) Give an example of closed, disjoint subsets \( A \) and \( B \) of the plane \( \mathbb{R}^2 \) for which \( d(A, B) = 0 \).

(iii) If \( A \) and \( B \) are closed and disjoint, show that there are open sets \( U \) and \( V \) with \( A \subset U \), \( B \subset V \), and \( U \cap V = \emptyset \).

(iv) If \( A \) is compact, \( B \) is closed, and \( A \) and \( B \) are disjoint, show that \( d(A, B) \) is nonzero.