MAA 4226 Advanced Calculus **Homework Set 4** **Bowers Fall 2000**

1. The **identity function** $\mathrm{id}_X: X \to X$ on the set X is defined by $\mathrm{id}_X(x) = x$ for all $x \in X$. Let $f: X \to Y$ be a function. Prove that f is a bijection if and only if there exists $g: Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

2. Define $f: \mathbf{N} \to \mathbf{Z}^* = \{\alpha \in \mathbf{Z}: \alpha \neq 0\}$ by

$$f(\alpha) = (-1)^\alpha \left\lfloor \frac{\alpha + 1}{2} \right\rfloor.$$

Note that $f(\alpha) = \alpha/2$ if α is even and $f(\alpha) = -(\alpha+1)/2$ if α is odd. Prove that f is a bijection by exhibiting a function $g: \mathbf{Z}^* \to \mathbf{N}$ for which $g \circ f = \mathrm{id}_{\mathbf{N}}$ and $f \circ g = \mathrm{id}_{\mathbf{Z}^*}$.

3. This exercise gives an explicit formula for a bijection from $\mathbf{N}$ to $\mathbf{Q}^+$, the set of positive rationals. It is based on the unique factorization of integers into primes. Let $f: \mathbf{N} \to \mathbf{Z}^*$ be any bijection (for example, the one of the previous exercise). For $n \in \mathbf{N}$, let $n = 1p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ where $p_1, p_2, \cdots p_s$ are distinct primes, $\alpha_i \in \mathbf{N}$, and $s \geq 0$ is an integer ($s = 0$ means that $n = 1$). Define

$$g(n) = 1p_1^{f(\alpha_1)} \cdots p_s^{f(\alpha_s)} \in \mathbf{Q}^+.$$

By uniqueness of prime decompositions, $g(n)$ defines a function $g: \mathbf{N} \to \mathbf{Q}^+$. Notice that $g(1) = 1$. Prove that g is a bijection. Hint: Define $h: \mathbf{Q}^+ \to \mathbf{N}$ by the following process. For $r \in \mathbf{Q}^+$, write $r = a/b$ where $a, b \in \mathbf{N}$ are relatively prime integers. Write $a = 1p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, $b = 1q_1^{\beta_1} \cdots q_t^{\beta_t}$, and note that $p_i \neq q_j$ for all i and j. Define

$$h(r) = 1p_1^{f^{-1}(\alpha_1)} \cdots p_s^{f^{-1}(\alpha_s)} q_1^{f^{-1}(-\beta_1)} \cdots q_t^{f^{-1}(-\beta_t)}.$$

Prove that $h \circ g = \mathrm{id}_{\mathbf{N}}$ and $g \circ h = \mathrm{id}_{\mathbf{Q}^+}$.

4. Let X, d be a metric space. For each $x \in X$ and nonempty subsets A and B of X, define

$$d(x, A) = \inf\{d(x, a): a \in A\}$$
$$d(A, B) = \inf\{d(a, b): a \in A, b \in B\}.$$

(i) Prove that $d(x, A) = 0 \Leftrightarrow x \in \overline{A}$.

(ii) Give an example of closed, disjoint subsets A and B of the plane $\mathbf{R}^2$ for which $d(A, B) = 0$.

(iii) If A and B are closed and disjoint, show that there are open sets U and V with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

(iv) If A is compact, B is closed, and A and B are disjoint, show that $d(A, B)$ is nonzero.