MAS 3301 Modern Algebra Homework Set 1

- 1. Prove that $\sqrt{3}$ is irrational. You will use the property that if a^2 is a multiple of 3, then a itself must be a multiple of 3. Explain why this is so.
- 2. Prove that $\sqrt{5}$ is irrational.

The Field Axioms:

Let **F** be a set with two binary operations + and \times that satisfy:

FA0: (closure) For all a, b in \mathbf{F} , a + b and $a \times b$ are in \mathbf{F} .

FA1: (commutative) For all a, b in $\mathbf{F}, a + b = b + a$ and $a \times b = b \times a$.

FA2: (associative) For all a, b, c in $\mathbf{F}, a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$.

FA3: (identities) An additive identity 0 and a multiplicative identity 1 are in \mathbf{F} : for all a in \mathbf{F} , a + 0 = a and $a \times 1 = a$.

FA4: (inverses) For all a and nonzero b in **F**, there exists in **F** an additive inverse u for a and a multiplicative inverse v for b: a + u = 0 and $b \times v = 1$. Notation: u is denoted -a and v is denoted b^{-1} or $\frac{1}{b}$.

FA5: (distributive) For all a, b, c in \mathbf{F} , $a \times (b + c) = a \times b + a \times c$.

- 3. Which of the field axioms do the natural numbers **N**, the integers **Z**, and the rational numbers **Q** satisfy?
- 4. Let $\mathbf{F} = \{x + y\sqrt{5}: x, y \in \mathbf{Q}\}$ with the usual addition and multiplication operations. Since \mathbf{F} is a subset of the real numbers \mathbf{R} , FA1, FA2, and FA5 are automatically satisfied. Show that the remaining field axioms are satisfied so that \mathbf{F} is a field.
- 5. Read Chapter 1, pp. 1–27 and pp. 211–214 of Conway and Guy.
- 6. For complex numbers a and b, prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$

if either |a| = 1 or |b| = 1. What exception must be made if |a| = |b| = 1?