MAS 3301 Modern Algebra Homework Set 2

1. Simplify each expression, rewriting in the form \( a + b\, i \), where \( a \) and \( b \) are real numbers.

(i) \( \frac{1 + 2\, i}{3 - 4\, i} + \frac{2 - \, i}{5\, i} \) \hspace{1cm} \text{Ans.} \quad -\frac{2}{5}

(ii) \( \frac{5}{(1 - i)(2 - i)(3 - i)} \) \hspace{1cm} \text{Ans.} \quad \frac{1}{2}\, i

(iii) \( (1 - i)^4 \) \hspace{1cm} \text{Ans.} \quad -4

2. Verify that each of the two numbers \( z = 1 \pm i \) satisfies the equation \( z^2 - 2z + 2 = 0 \).

3. In each case, sketch in the complex plane the set of complex numbers determined by the given condition.

(i) \( |z - 1 + i| = 1 \); \hspace{1cm} (ii) \( |z + i| \leq 3 \); \hspace{1cm} (iii) \( \text{Re}(\overline{z} - iz) = 2 \); \hspace{1cm} (iv) \( |2z - i| = 4 \).

4. By setting \( z = a + bi \) and \( w = c + di \), verify by calculation that

(i) \( z \pm w = \overline{z} \pm \overline{w} \); \hspace{1cm} (ii) \( z\, w = \overline{z} \, \overline{w} \); \hspace{1cm} (iii) \( \left( \frac{z}{w} \right) = \overline{\frac{z}{w}} \).

5. By writing the individual factors on the left of each equation in polar form, performing the needed operations, and finally changing back to rectangular coordinates, verify that

(i) \( i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i) \); \hspace{1cm} (ii) \( 5i/(2 + i) = 1 + 2i \);

(iii) \( (-1 + i)^7 = -8(1 + i) \); \hspace{1cm} (iv) \( (1 + \sqrt{3}i)^{-10} = 2^{-11} (-1 + \sqrt{3}i) \).

6. Use de Moivre’s formula to derive the following trigonometric identities:

(i) \( \cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta \); \hspace{1cm} (ii) \( \sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta \).

7. If you have not done so already, work problem 6 from the first homework set.

8. Each of the following numbers is algebraic. For each, find a polynomial with integer coefficients that has that number as a root.

(i) \( \sqrt[3]{7}/3 \); \hspace{1cm} (ii) \( \sqrt[3]{7}/\sqrt{2} \); \hspace{1cm} (iii) \( 3\sqrt[3]{7} + 1 \); \hspace{1cm} (iv) \( \sqrt{2} + \sqrt{3} \); \hspace{1cm} (v) \( 2 + \sqrt{3} \).