MAS 3301 Modern Algebra Homework Set 2

1. Simplify each expression, rewriting in the form $a + bi$, where a and b are real numbers.

   (i) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$ *Ans.* $-\frac{2}{5}$

   (ii) $\dfrac{5}{(1-i)(2-i)(3-i)}$ *Ans.* $\frac{1}{2}i$

   (iii) $(1-i)^4$ *Ans.* -4

2. Verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$.

3. In each case, sketch in the complex plane the set of complex numbers determined by the given condition.

   (i) $|z - 1 + i| = 1$; (ii) $|z + i| \leq 3$; (iii) $\mathrm{Re}(\overline{z} - iz) = 2$; (iv) $|2z - i| = 4$.

4. By setting $z = a + bi$ and $w = c + di$, verify by calculation that

   (i) $\overline{z \pm w} = \overline{z} \pm \overline{w}$; (ii) $\overline{zw} = \overline{z}\ \overline{w}$; (iii) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.

5. By writing the individual factors on the left of each equation in polar form, performing the needed operations, and finally changing back to rectangular coordinates, verify that

   (i) $i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i)$; (ii) $5i/(2 + i) = 1 + 2i$;

   (iii) $(-1 + i)^7 = -8(1 + i)$; (iv) $(1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i)$.

6. Use de Moivre's formula to derive the following trigonometric identities:

   (i) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; (ii) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

7. If you have not done so already, work problem 6 from the first homework set.

8. Each of the following numbers is algebraic. For each, find a polynomial with integer coefficients that has that number as a root.

   (i) $\dfrac{\sqrt[3]{7}}{3}$; (ii) $\dfrac{\sqrt[3]{7}}{\sqrt{2}}$; (iii) $\sqrt[3]{7} + 1$; (iv) $\sqrt{2} + \sqrt{3}$; (v) $\sqrt{2} + \sqrt[3]{3}$.