MAS 3301 Modern Algebra Homework Set 3

Definitions. A set $\mathbf{F}$ is a *ring* if addition and multiplication operations are defined on $\mathbf{F}$ that satisfy all the field axioms except that multiplicative inverses are not required to exist for every element of $\mathbf{F}$. The standard example is the ring of integers $\mathbf{Z}$. A nonzero element a of $\mathbf{F}$ is a *zero divisor* if there is a nonzero element b for which $ab = 0$. The ring of integers has no zero divisors. Exercise 2 below gives an example of a ring that does have zero divisors.

1. Let $\mathbf{F}$ be a ring.

   (i) Show that if a is a zero divisor of $\mathbf{F}$, then a fails to have a multiplicative inverse.

   (ii) Show that a field has no zero divisors.

2. Recall that the dual numbers are exactly like the complex numbers except that $i^2 = 1$. The dual numbers satisfy all the field axioms except that multiplicative inverses sometimes fail to exist. The multiplicative identity for the dual numbers is, as for the complex numbers, $1 = 1 + 0i$. The dual numbers do form a ring.

   (i) Find the multiplicative inverse of $1+2i$ (which does exist), and show that $1+i$ does not have a multiplicative inverse (Hint: for the second assertion use exercise 1(i)).

   (ii) Characterize those dual numbers $a + bi$ that do have a multiplicative inverse, ie, find conditions on a and b that determine whether or not $(a + bi)^{-1}$ exists.

   (iii) Find a general formula for $(a + bi)^{-1}$ when it exists.

3. The double numbers are defined exactly like the complex and dual numbers except that $i^2 = 0$. Again, $1 = 1 + 0i$ is the multiplicative identity.

   (i) Find the multiplicative inverses of $1+2i$ and $1+i$ (both exist). Show that i^{-1} fails to exist.

   (ii) Characterize those double numbers $a + bi$ that do have a multiplicative inverse, ie, find conditions on a and b that determine whether or not $(a + bi)^{-1}$ exists.

   (iii) Find a general formula for $(a + bi)^{-1}$ when it exists.

4. Let $\mathbf{D} = \{(a, b)\colon a, b \in \mathbf{R}\}$ with $(a, b)+(c, d) = (a+c, b+d)$ and $(a, b)\times(c, d) = (ac^2, bd^2)$.

   (i) Is $\times$ commutative? associative?

   (ii) The element I of $\mathbf{D}$ is a *right identity* if $z \times I = z$ for all z in $\mathbf{D}$ and a *left identity* if $I \times z = z$ for every z in $\mathbf{D}$. Find all the right identities in $\mathbf{D}$ (there are several).

   (iii) Are there any left identities in $\mathbf{D}$? Why or why not?