MAS 3301 Modern Algebra Homework Set 4

1. Let $\mathbf{F} = (0, \infty) = \{x \in \mathbf{R}: x > 0\}$, the set of positive real numbers. We use the symbols $\oplus$ and $\otimes$ to denote the following operations on $\mathbf{F}$: for $a, b \in \mathbf{F}$, $a \oplus b = ab$ and $a \otimes b = a^{\ln b}$. For items (i) and (ii) below, you might find useful the calculus formulas $a^x = e^{x \ln a}$, $e^{\ln x} = x$, $\ln e^x = x$, and the laws of exponents and logarithms.

   (i) Show that $\mathbf{F}$ is a field under $\oplus$-addition and $\otimes$-multiplication.

   (ii) Define $h: \mathbf{R} \to \mathbf{F}$ by $h(x) = e^x$. Show that for any two reals $a, b \in \mathbf{R}$,

   $$h(a + b) = h(a) \oplus h(b) \qquad \text{and} \qquad h(ab) = h(a) \otimes h(b).$$

2. Let $\mathbf{H}$ denote the quaternions and recall that a typical quaternion q can be written $q = q_0 + q_1 i + q_2 j + q_3 k$ where the q_n's are real numbers. The q_0 term is the *real part* of q and the term $\vec{q} = q_1 i + q_2 j + q_3 k$ is the *vector part* of q, so we have $q = q_0 + \vec{q}$. If $q_0 = 0$, $q = \vec{q}$ is then called a *pure* quaternion and is a vector in $\mathbf{R}^3$. Multiplication of quaternions is performed using the usual associative and distributive laws, but the commutative law fails to hold. The unit vectors $i = \hat{\imath}, j = \hat{\jmath}, k = \hat{k}$ multiply as follows: $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$.

   (i) Show that for any pure quaternion q, $q^2 = -(q_1^2 + q_2^2 + q_3^2)$.

   (ii) Two pure quaternions p and q are said to be *orthogonal* if their dot product $p \cdot q = \vec{p} \cdot \vec{q} = p_1 q_1 + p_2 q_2 + p_3 q_3$ is equal to 0. Show that if p and q are orthogonal, then $pq = -qp$.

   (iii) Show that every quaternion q can be written in the form $z + wj$ for some complex numbers z and w.

   (iv) Show that every quaternion q can be written in the form $u + jv$ for some complex numbers u and v.

   (v) If $z + wj = q = u + jv$ as in (iii) and (iv), what is the relationship between z and u and between w and v?

3. Simplify each expression, writing each quaternion in the standard form $q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + \vec{q}$.

   (i) $\dfrac{7 - i}{j + k}$; (ii) $(1 + i)(1 + j) - (1 + j)(1 + i)$; (iii) $\dfrac{1}{i + j + k}$

4. Recall that the *dot product* of arbitrary quaternions p and q is given by $p \cdot q = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3$, and the *cross product* of pure quaternions $p = \vec{p}$ and $q = \vec{q}$ is given by $p \times q = (p_2 q_3 - p_3 q_2) i + (p_3 q_1 - p_1 q_3) j + (p_1 q_2 - p_2 q_1) k$.

   (i) Show that for any two quaternions p and q, the dot product can be written as $p \cdot q = \frac{1}{2}(\overline{p} q + \overline{q} p)$.

   (ii) Show that for any two pure quaternions $p = \vec{p}$ and $q = \vec{q}$, the cross product can be written as $p \times q = pq + p \cdot q$.