MAS 3301 Modern Algebra Homework Set 5

1. Construct Cayley tables for addition and multiplication in $\mathbf{Z}/4$, $\mathbf{Z}/5$, and $\mathbf{Z}/8$.

2. How, by just looking at a Cayley table for multiplication, does one determine if an element has a multiplicative inverse? Which of the rings of modular integers from Problem 1 are fields?

3. For what values of n does $\mathbf{Z}/n$ have zero divisors? Why? Explain fully.

4. We have already seen that if a is a zero divisor in a ring $\mathbf{D}$, then a does not have a multiplicative inverse (therefore, $\mathbf{D}$ is not a field). The converse of this fact is the statement

   If the element a of the ring $\mathbf{D}$ does not have a multiplicative inverse, then a must be a zero divisor.

   (i) Give an example of a ring where this converse statement is false. HINT: Look at the most natural ring around!

   (ii) Prove that if $\mathbf{D}$ is a finite ring, ie, $\mathbf{D}$ has only finitely many elements, then this converse statement is true.

   (iii) From Problem 3 and item (ii) above, for which n is $\mathbf{Z}/n$ a field?

5. Find a field with four elements. HINT: Call such a field $\mathbf{F}$ and let 0 and 1 denote the additive and multiplicative inverses. Call the remaining elements of $\mathbf{F}$ a and b so that $\mathbf{F} = \{0, 1, a, b\}$. Use the field axioms to fill in as much of the Cayley tables for addition and multiplication as you can and work with equations to see if the whole table is forced on you.

6. Write each of the following elements of $\mathbf{Z}/13$ as $\overline{a}$ for some $a \in \{0, 1, \ldots, 12\}$.

   (i) $-\overline{7}$ (ii) $\overline{10}^{-1}$ (iii) $\overline{4}^2 - \overline{6}$ (iv) $(\overline{131})^{10}$ (v) $-\overline{14}$

7. Which elements of $\mathbf{Z}/7$ have 4th roots? Find all 4th roots of elements that have them.

8. Find all the square roots of $\overline{4}$ in $\mathbf{Z}/5$ and $\mathbf{Z}/7$.

9. Find the sum of all the elements of $\mathbf{Z}/p$ whenever p is prime, ie, find $\overline{0} + \overline{1} + \cdots + \overline{p-1}$. HINT: Experiment with various small primes p.

10. Solve the equation $x^2 + \overline{2}x - \overline{3} = \overline{0}$ in $\mathbf{Z}/5$. Does it make sense to apply the quadratic formula? Now solve the equation $\overline{2}x^2 + \overline{1} = \overline{0}$ in $\mathbf{Z}/5$. What happened?