MAS 3301 Modern Algebra Homework Set 7

Recall that a group G is a set of elements with a binary operation $\star$ that satisfies the following *group axioms*:

GA0: **Closure.** For each $a, b \in G$, $a \star b \in G$;

GA1: **Associative.** For each $a, b, c \in G$, $a \star (b \star c) = (a \star b) \star c$;

GA2: **Identity.** There exists a $\star$-identity, an element $e \in G$ such that $a \star e = e \star a = a$, for all $a \in G$;

GA3: **Inverses.** For each $a \in G$, there exists a $\star$-inverse, an element $b \in G$ such that $a \star b = b \star a = e$.

The *order* of the group G, denoted $o(G)$, is the number of elements in G if G is finite, and equal to ∞ otherwise. The *order* of the element a of G is the least positive integer $o(a) = n$ for which the n-fold $\star$-product $a^n = a \star \cdots \star a$ is equal to the identity e, if such n exists; otherwise $o(a) = \infty$. Note that $o(e) = 1$. G is *cyclic* if there is at least one element g in G such that every other element a of G is a power of g, ie, if for each $a \in G$, there exists $k \in \mathbf{Z}$ such that $a = g^k$. In this case, g is called a *generator* of G.

1. Find the order of the element a of the group G.

   (i) $a = \overline{3}$, $G = \mathbf{Z}/10, +$

   (ii) $a = \overline{3}$, $G = (\mathbf{Z}/10)^*, \times$

   (iii) $a = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$, $G = \mathrm{GL}(2, \mathbf{R})$

2. Which of the following groups are cyclic? For any that are, name all the generators.

   $\mathbf{Z}/4, +\quad (\mathbf{Z}/5)^*, \times\quad \mathbf{F}_4, +\quad \{\pm 1, \pm i\}, \times\quad (\mathbf{Z}/10)^*, \times\quad \mathbf{Z}/2 \oplus \mathbf{Z}/2, +\quad (\mathbf{Z}/8)^*, \times$

3. All the groups listed in Problem 2 have order 4. Which are isomorphic to which? Describe explicit isomorphisms.

4. What is the order of the group $(\mathbf{Z}/100)^*, \times$? What is the order of the element $\overline{7}$ in $(\mathbf{Z}/100)^*, \times$?

5. What are the possible orders of the elements of $(\mathbf{Z}/100)^*, \times$? Answer the same question for $\mathbf{Z}/100, +$.

6. Let $G, \times$ and $H, \times$ be multiplicative groups. Define an operation $\star$ on $G \times H = \{(g, h) \colon g \in G, h \in H\}$ by $(g, h) \star (g', h') = (gg', hh')$. Verify that $G \times H, \star$ is a group. What is the order of $G \times H$ in terms of $o(G)$ and $o(H)$?