MAS 3301 Modern Algebra Homework Set 7

Recall that a group $G$ is a set of elements with a binary operation $\ast$ that satisfies the following group axioms:

GA0: **Closure.** For each $a, b \in G$, $a \ast b \in G$;

GA1: **Associative.** For each $a, b, c \in G$, $a \ast (b \ast c) = (a \ast b) \ast c$;

GA2: **Identity.** There exists an identity element $e \in G$ such that $a \ast e = e \ast a = a$, for all $a \in G$;

GA3: **Inverses.** For each $a \in G$, there exists an inverse element $b \in G$ such that $a \ast b = b \ast a = e$.

The **order** of the group $G$, denoted $o(G)$, is the number of elements in $G$ if $G$ is finite, and equal to $\infty$ otherwise. The order of the element $a$ of $G$ is the least positive integer $o(a)$ for which the $n$-fold $\ast$-product $a^n = a \ast \cdots \ast a$ is equal to the identity $e$, if such $n$ exists; otherwise $o(a) = \infty$. Note that $o(e) = 1$. $G$ is **cyclic** if there is at least one element $g$ in $G$ such that every other element $a$ of $G$ is a power of $g$, ie, if for each $a \in G$, there exists $k \in \mathbb{Z}$ such that $a = g^k$. In this case, $g$ is called a generator of $G$.

1. Find the order of the element $a$ of the group $G$.
   
   (i) $a = \bar{3}$, $G = \mathbb{Z}/10$, +

   (ii) $a = \bar{3}$, $G = (\mathbb{Z}/10)^*$, $\times$

   (iii) $a = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$, $G = \text{GL}(2, \mathbb{R})$

2. Which of the following groups are cyclic? For any that are, name all the generators.

   $\mathbb{Z}/4, +$ \quad $(\mathbb{Z}/5)^*, \times$ \quad $\mathbb{F}_4, +$ \quad $\{\pm 1, \pm i\}, \times$ \quad $(\mathbb{Z}/10)^*, \times$ \quad $\mathbb{Z}/2 \oplus \mathbb{Z}/2, +$ \quad $(\mathbb{Z}/8)^*, \times$

3. All the groups listed in Problem 2 have order 4. Which are isomorphic to which? Describe explicit isomorphisms.

4. What is the order of the group $(\mathbb{Z}/100)^*, \times$? What is the order of the element $\bar{7}$ in $(\mathbb{Z}/100)^*, \times$?

5. What are the possible orders of the elements of $(\mathbb{Z}/100)^*, \times$? Answer the same question for $\mathbb{Z}/100, +$.

6. Let $G, \times$ and $H, \times$ be multiplicative groups. Define an operation $\ast$ on $G \times H = \{(g, h) : g \in G, h \in H\}$ by $(g, h) \ast (g', h') = (gg', hh')$. Verify that $G \times H, \ast$ is a group. What is the order of $G \times H$ in terms of $o(G)$ and $o(H)$?