MAS 3301 Modern Algebra Homework Set 8

1. Using Lagrange's Theorem, explain why every group of prime order is cyclic.

2. Describe all isomorphisms from $\mathbf{Z} \oplus \mathbf{Z}, +$ to the set $S = \{3^p 5^q : p, q \in \mathbf{Z}\}, \times$.

3. Identify the group of units in the ring $\mathbf{Z}/5 \oplus \mathbf{Z}/6$ and write a Cayley Table for the operation.

4. The group of Problem 3 has order 8. Which of the following groups is it isomorphic to?

   $\mathbf{Z}/8, +$ $\quad\quad$ $\mathbf{Z}/2 \oplus \mathbf{Z}/4, +$ $\quad\quad$ $\mathbf{Z}/2 \oplus \mathbf{Z}/2 \oplus \mathbf{Z}/2, +$ $\quad\quad$ $\{\pm 1, \pm i, \pm j, \pm k\}, \times$

5. Let $G, \times$ be a group in which every nonidentity element has order 2. Prove that G is abelian.

6. When is $\mathbf{Z}/m \oplus \mathbf{Z}/n, +$ isomorphic to $\mathbf{Z}/mn, +$? Experiment with small values of m and n and note that $\mathbf{Z}/mn, +$ is cyclic, so the question that is equivalent is: when is $\mathbf{Z}/m \oplus \mathbf{Z}/n, +$ cyclic?

7. If $G, \times$ is a group such that $(ab)^2 = a^2 b^2$ for all $a, b \in G$, show that G must be abelian.

8. CHALLENGE PROBLEM: Prove that if the finite group $G, \times$ has even order, then there is an element $a \in G$ of order 2.