Belief Function Theory Applied to US Stock Market

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Belief Function Theory

Consider the following results of an election survey:

Social Democrats (S)	Labor (L)	Conservatives (R)	S or L	Undecided
40%	10%	30%	5%	15%

1. **Frame of discernment** is a nonempty finite set Θ of mutually exclusive and exhaustive outcomes. In this case, $\Theta = \{S, L, R\}$.
2. **Mass function** over Θ is a function $m : 2^{\Theta} \longrightarrow [0,1]$ that assigns a mass value to each $A \subseteq \Theta$ such that:

$$(1)\ m(\emptyset) = 0, \quad (2) \sum_{A\subseteq\Theta} m(A) = 1.$$

Mass values for the election example:

A	$\{S\}$	$\{L\}$	$\{R\}$	$\{S,L\}$	$\{S,L,R\}$	otherwise
$m(A)$	0.4	0.1	0.3	0.05	0.15	0

- 1 Those subsets with positive mass values are called **focal sets**. The collection of the focal sets is denoted by $\mathcal{F}_m$. Here,

$$\mathcal{F}_m = \{\{S\}, \{L\}, \{R\}, \{S,L\}, \{S,L,R\}\}.$$

- 2 $(\mathcal{F}_m, m)$ is called a **body of evidence**.

3. **Belief function** is a function $bel : 2^{\Theta} \longrightarrow [0,1]$, corresponding to a specific mass function m, that assigns to every proposition $A \subseteq \Theta$ the total amount of belief committed to that proposition, including all the subsets of A:

$$bel(A) = \sum_{B\subseteq A} m(B).$$

Therefore, the corresponding belief function for the election example can easily be obtained. For instance:

$$bel(\{S,L\}) = m(\{S\}) + m(\{L\}) + m(\{S,L\}) = 0.4 + 0.1 + 0.05 = 0.55.$$

4. **Plausibility function** is denoted by pl. $pl(A)$ gives the maximum amount of belief that could be assigned to the proposition A, if all evidence collected in the future supports the proposition.

- 1 In our election example the current belief for $\{S,R\}$ is 0.7. In the future, if the undecideds with mass value 0.15 support $\{S\}$, $\{R\}$ or $\{S,R\}$, then the belief for $\{S,R\}$ would increase to $0.7 + 0.15 = 0.85$. If moreover, those who currently support $\{S,L\}$, with mass value 0.05, move to support $\{S\}$ only, then the total belief for $\{S,R\}$ would increase to $0.7 + 0.15 + 0.05 = 0.9$.
- 2 In other words, the plausibility of the proposition $\{S,R\}$ is the maximum amount of belief that could be assigned to a proposition, which is 1, minus the amount of belief for its complement, $\{L\}$, i.e.,

$$pl(\{S,R\}) = 1 - bel(\overline{\{S,R\}}) = 1 - bel(\{L\}) = 1 - 0.1 = 0.9.$$

- 3 Here is the formal definition:

$$pl : 2^{\Theta} \longrightarrow [0,1]$$
$$pl(A) = 1 - bel(\bar{A}).$$

5. **What will happen to these functions when there is no uncertainty?**

- 1 The focal sets are the singletons.
- 2 Belief and plausibility functions become identical, and turn into a probability measure.

Dempster's Rule of Combination

Assume, another election survey is done and we have:

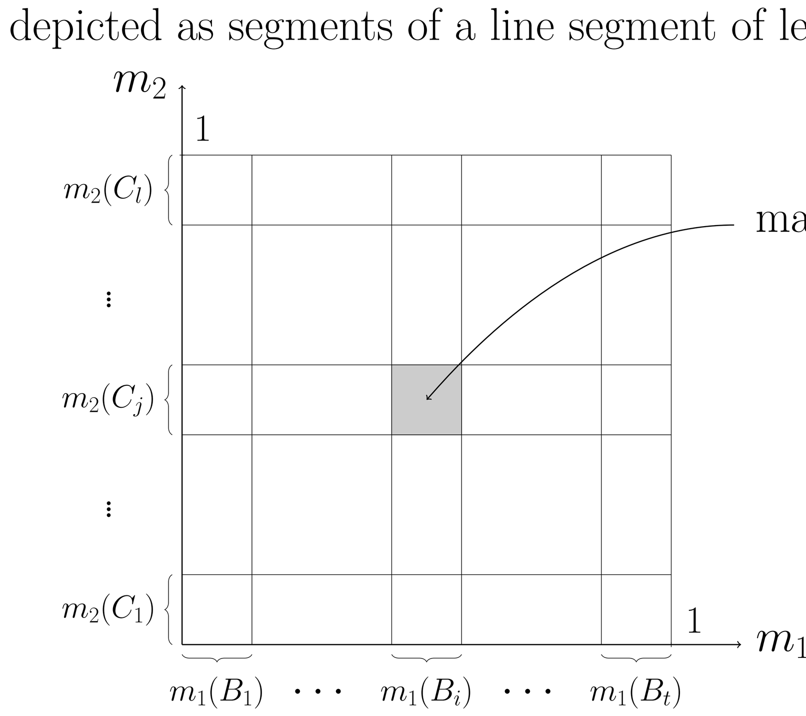
Social Democrats (S)	Labor (L)	Conservatives (R)	Undecided
30%	10%	25%	35%

Thus, the mass values are assigned as follows:

A	$\{S\}$	$\{L\}$	$\{R\}$	$\{S,L,R\}$	otherwise
$m(A)$	0.3	0.1	0.25	0.35	0

1. **Can we take these two surveys, and merge them, in a sense, to obtain their sum?** *Dempster* first introduced a method to combine evidence coming from different sources. Let us illustrate the idea for the case of two bodies of evidence:

- 1 Suppose $(\mathcal{F}_{m_1}, m_1)$ and $(\mathcal{F}_{m_2}, m_2)$ are two distinct bodies of evidence which are defined over the same frame of discernment Θ.
- 2 Let $\mathcal{F}_{m_1} = \{B_1, \ldots, B_t\}$. Since the mass values add up to 1, $m_1(B_i)$, $1 \le i \le t$, can be depicted as segments of a line segment of length 1. Similarly, if $\mathcal{F}_{m_2} = \{C_1, \ldots, C_l\}$, then the mass values $m_2(C_j)$, $1 \le j \le l$, can also be depicted as segments of a line segment of length 1:



- 3 Therefore, the total mass value committed to A is $\sum_{B_i\cap C_j = A} m_1(B_i)m_2(C_j)$.
- 4 Here, we should discard all the rectangles that are committed to the empty set, since we need the mass value of the empty set to be zero. Moreover, we want a normalization constant so that the new mass values add up to 1:

$$k := \Big(1 - \sum_{B_i\cap C_j=\emptyset} m_1(B_i)m_2(C_j)\Big)^{-1} = \Big(\sum_{B_i\cap C_j\neq\emptyset} m_1(B_i)m_2(C_j)\Big)^{-1}.$$

2. The mass function $m_{1\oplus 2} = m_1 \oplus m_2$, combined according to Dempster's rule $\oplus$, is defined as

$$m_{1\oplus 2}(A) = \begin{cases} k \sum_{B\cap C = A} m_1(B)m_2(C) & if\ A \neq \emptyset, \\ 0 & if\ A = \emptyset. \end{cases}$$

3. Let us get back to our election example. The combined mass function is:

A	$\{S\}$	$\{L\}$	$\{R\}$	$\{S,L\}$	$\{S,L,R\}$	otherwise
$m_{1\oplus 2}(A)$	0.4758	0.0967	0.3234	0.026	0.0781	0

4. Dempster's rule becomes computationally intractable as the number of bodies of evidence grows. **Monte Carlo methods** can be used in these cases.

Application to US Stock Market

1. **What is the most promising industry in the US in the next month?**

- 1 There are 12 industries in the US stock Market which form the frame of discernment Θ.
- 2 Collect *analyst ratings* from `http://www.MarketBeat.com`.
- 3 Each analyst rating form a mass function. Combine them using Dempster's rule.
- 4 Pick the industry with the highest mass value, and invest 1 dollar.
- 5 Sell the stocks 25 business days later and calculate the excess return.

2. **What are the most promising stocks in the next month?**

- 1 To find an answer (portfolio), repeat the above procedure with the modified *frame of discernment* and *mass functions*.
- 2 On each business day, choose top x% of the stocks with the highest mass value to invest.

3. Repeat the process for each business day from 2/5/2015 to 10/17/2016. Finally, take the average to find the average of the excess returns on a day.
4. **Results:** The average of excess returns are depicted when n analysts ratings are considered ($n = 5, 10$).

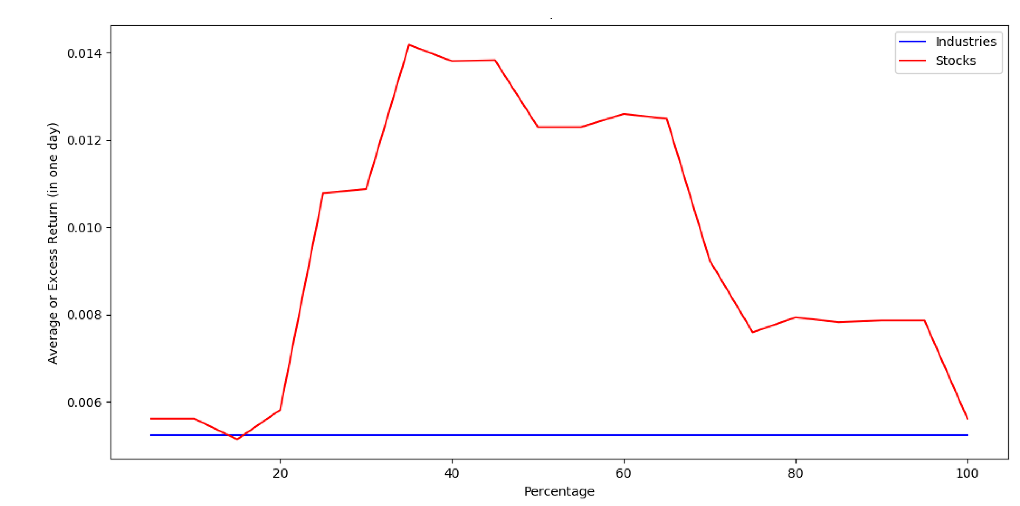


Figure 1: Average of excess returns when $n = 5$.

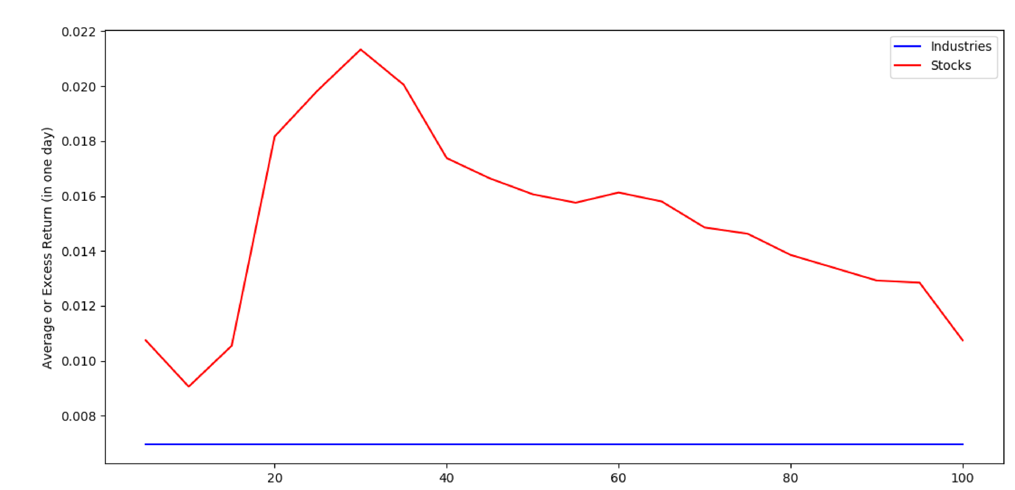


Figure 2: Average of excess returns when $n = 10$.

References

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