Octonions – Hilbert Spaces, Fibrations and Analysis (Take II)

Ben Prather

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FSU Department of Mathematics

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Retest

Original defense attempt on June 10th 2021.

The previous document had confusion regarding various notions of a subspace generated by a set and lacked any development of linear maps between modules.

Today we will go over the significant research advances since then.

Motivation

What role does pure mathematics have,
if not to develop the tools needed by other fields?

> *Nobody has managed to develop a good theory of octonionic linear algerba. – John Baez*

This is based on two gaps:

- the lack of a definition of modules and
- linear maps between modules.

The octonionic Hilbert spaces address the first,
while elementary row operations and automorphisms address the latter.

Octonions

The octonions are an algebra generated on $\mathbb{R}^8$ generated by the following basis:

e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Note that $(e_1 e_2)e_4 = e_7 = -e_1(e_2 e_4)$.

Octonions - Properties

- Let $q = \sum_{i=0}^{7} q_i e_i$ and q's conjugate be $\overline{q} = q_0 e_0 - \sum_{i=1}^{7} q_i e_i$.
- **Norm**: $N(q) = q\overline{q} = \overline{q}q = \sum_{i=0}^{7} q_i^2$.
- **Inverses**: $q^{-1} = \frac{1}{N(q)}\overline{q}$
- **Associator**: $[p, q, r] = (pq)r - p(qr)$.
- **Alternative**: $[p, q, r] = [q, r, p] = -[q, p, r]$.
- **Composition**: $N(pq) = N(p)N(q)$.

Theorem (1.2.1, Hurwitz)

The only unital division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

Algebras

But what is an algebra?

An algebra is usually defined to be associative.

Schafer defines algebras explicitly, omitting associativity.

I follow the example of pre-measures, pre-sheaves etc. and introduce pre-algebras to describe non-associative algebras.

Pre-Algebras

Definition (1.1.3)

Let R be a commutative ring with identity.
A **pre-R-algebra**, A, is an R-module with a bilinear product $\cdot : A \times A \to A$ called multiplication.

Bilinearity gives us, for any x, y, z in A and a and b in R :

- Left distributive: $x \cdot (y + z) = x \cdot y + x \cdot z$.
- Right distributive: $(x + y) \cdot z = x \cdot z + y \cdot z$.
- Scalar compatibility: $(ax) \cdot (by) = (ab)(x \cdot y)$.

A **unital** pre-R-algebra has a 1_A such that for all x we have $1_A \cdot x = x \cdot 1_A = x$.

An R-algebra is then an associative pre-R-algebra.

Alternative Pre-Algebras

A pre-R-algebra is **alternative** if $[x, x, y] = [x, y, x] = [y, x, x] = 0$.

A **division** pre-R-algebra guarantees that for any two non-zero x and y, there exist s and t such that $y = s \cdot x$ and $y = x \cdot t$.

If $y = 1_A$ we call s and t the right and left inverse respectively. If $s = t$ we denote this by x^{-1}.

Without associativity it may be that $x^{-1}(xy) \neq y$.

Alternativity is sufficient to guarantee $x^{-1}(xy) = y$.

Composition Pre-Algebras

Definition (Section 1.2)

A **composition pre-algebra** is a pre-R-algebra with a non-degenerate quadratic form $N(a)$ that is multiplicative. Thus for all a and b in A $N(ab) = N(a)N(b)$.

Over the real numbers there are seven composition pre-algebras:

The positive definite $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, plus the indefinite $\mathbb{C}^-$, $\mathbb{H}^-$, and $\mathbb{O}^-$.

The **split-octonions**, $\mathbb{O}^-$, can be found within the complexified octonions.

Pre-Modules

Let R be a commutative ring with unity.
Let A be a pre-R-algebra, and X an R-module.
Let $\mathrm{End}_R(X)$ be the linear maps from X to itself, as an R-module.
Let $\varphi : A \to \mathrm{End}_R(X)$, and the enveloping algebra $\overline{\varphi_A}$ be the closure of the image of A in $\mathrm{End}_R(X)$.

Definition (2.2.1)

A **left pre-module**, X, over a pre-R-algebra A is an R-module and linear map $\varphi : A \to \mathrm{End}_R(X)$ such that X is a left $\overline{\varphi(A)}$-module.

For r in $\overline{\varphi_A}$ and x in X, let $rx = r \cdot x = r(x)$.
For a in A let $ax = a \cdot X = \varphi_a(x)$.

Free Pre-Modules

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Let X be a pre-A-module and $E \subset X$.

An element $x \in X$ is **$\overline{\varphi_A}$-linearly dependent** on E if there are is a finite subset of distinct e_i in E and elements of $\overline{\varphi_A}$, a_i, such that: $x = \sum_{i=1}^{n} a_i \cdot e_i$.

The $\mathrm{Span}_{\overline{\varphi_A}}(E)$ is the set of all $x \in X$ that are linearly dependent on E.

A set E is **$\overline{\varphi_A}$-linearly independent** if no $e_i \in E$ is linearly dependent on $E - \{e_i\}$.

X is a **$\overline{\varphi_A}$-free pre-module** if it has a linearly independent generating set, or basis.

Examples

Any pre-algebra acting on itself by left multiplication.

Functions from any set to an pre-algebra with multiplication defined point-wise.

For finite sets, this yields A^n.

Proposition (2.3.3, P.)

The pre-A-modules A^n are $\overline{\varphi_A}$-free.

The quaternions acting on the octonions as a sub-algebra yield a non trivial example.

Results for $\mathbb{O}^n$

Proposition (2.3.4, P.)

The subspaces of $\mathbb{O}^n$ may have $\overline{\varphi_{\mathbb{O}}}$-bases with different cardinality.

Proposition (2.3.5, P.)

Let $X = \mathbb{O}^n$, and $x \in X$. Using elementary row operations and $\mathrm{Aut}(\mathbb{O})$ we can transform x to one of the following cases:

- $(\cos\theta + \sin\theta e_1, e_2, \cos\phi e_3 + \sin\phi e_4, \ldots)$,
- $(\cos\theta + \sin\theta e_1, e_2, e_3, e_4, \ldots)$, *and*
- $(1, e_1, e_2, \cos\phi e_3 + \sin\phi e_4, \ldots)$,
- $(1, e_1, e_2, e_3, e_4, \ldots)$,

with the remaining components of x in $\mathrm{Span}_{\mathbb{R}}(\{e_5, e_6, e_7\})$.

Notes on Proposition 2.3.5

Elementary row operations are in $\mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n) \cong \mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$.
A slight modification of Gram-Schmidt allows us to make the first component contain any real coefficient, with the entire set mutually orthogonal.

Elements of $\mathrm{Aut}(\mathbb{O})$ can be found to move the first few components of x to the subspace spanned by e_1 and e_2.

A similar application of Gram-Schmidt allows us to isolate e_3, as we had done with the real coefficient.

Finally, an element of $\mathrm{Aut}(\mathbb{O})$ preserving e_1 and e_2, while moving the next available component of x to e_4.

This fixes e_5, e_6 and e_7.
In passing, d is defined as the rank of the form.

Hermitian Spaces

Let R be a commutative ring with unity.

Definition (2.4.1)

Let A be a $*$-pre-algebra over R.
A **Hermitian space** is a left pre-A-module with a **Hermitian form** $\langle \cdot, \cdot \rangle : X \times X \to A$ such that for all x, y and z in X and a and b in R:

$\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$

x and y are **orthogonal** if $\langle x, y \rangle = 0$.

If S is a subset of X the **orthogonal complement** of S is $S^{\perp} = \{x \in X | \langle x, s \rangle = 0 \text{ for all } s \in S\}$.

Orthogonal Closure

The $S^{\perp\perp}$ is the **orthogonal closure** of S.

A set S is **closed** if $S = S^{\perp\perp}$.

If B is a set of mutually orthogonal vectors and $S = B^{\perp\perp}$, then B is an **orthognal basis** for S.

S is **cyclic** if $S = \{x\}^{\perp\perp}$ for some $x \in X$.

Examples

Let A be any composition pre-algebra and S any finite set.
Let X be the functions from S to A with the action defined point-wise by left multiplication.
Then $\langle x, y \rangle = \sum_{s \in S} x_s \overline{y_s}$ yields a Hermitian form.

If S is infinite convergence becomes an issue.
We can define X by restricting the functions to those with only finitely many non-zero values.

If A is a composition pre-algebra over $\mathbb{R}$ or $\mathbb{C}$ we can use analytical tools to define convergence.
If S is $\mathbb{N}$ this yields the l^2 **sequence space**.
If S is measurable this produces L^2 **spaces**.

Results for $\mathbb{O}^n$

Proposition (2.4.9, P.)

The transformations of Proposition 2.3.5 satisfy
$\phi(S)^{\perp\perp} = \phi(S^{\perp\perp})$.
For $\psi \in \mathrm{End}_{\mathbb{O}|\mathbb{R}}(\mathbb{O}^n)$, $\psi(S)^{\perp} = \psi^{-1}(S^{\perp})$.
For $\alpha \in \mathrm{Aut}\,\mathbb{O}$, $\alpha(S)^{\perp} = \alpha(S^{\perp})$.

Theorem (2.4.10, P.)

Using strong orthogonality, the cycles in $\mathbb{O}^n$, $\{x\}^{\perp\perp}$, *are the set* $\{qx | q \in A\}$, *where*

- $A = \mathbb{O}$ *if* $d = 1$,
- $A \cong \mathbb{C}$, *where* A *is generated by* $\{\overline{x_i} x_j\}$ *if* $d = 2$, *or*
- $A = \mathbb{R}$ *if* $d > 2$,

where d *is defined as in Propositions 2.3.5.*

Notes on Theorem 2.4.10

From Proposition 2.3.5, let $x = (x_1, ..., x_d, 0, ..., 0)$.
For $y \in \{x\}^\perp$ and $z \in \{x\}^{\perp\perp}$,
let $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_d, 0, ..., 0)$.

If $d = 2$, then $x_1\overline{y_1} + x_2\overline{y_2} = 0$, so $\overline{y_1} = -\frac{1}{N(x_1)}\overline{x_1}(x_2\overline{y_2})$.

Similarly, $z_1\overline{y_1} + z_2\overline{y_2} = 0$ so $z_2 = -\frac{1}{N(y_2)}(z_1\overline{y_1})y_2$.

Hence,

$$z_2 = \frac{1}{N(x_1)N(y_2)}(z_1(\overline{x_1}(x_2\overline{y_2})))y_2. \qquad (1)$$

Since x is given, (1) imposes restrictions on z_1.
Specifically, z_2 must remain constant for any $y_2 \in \mathbb{O} - \{0\}$.

Notes on Theorem 2.4.10

Thus we find y_2 and y_2' and evaluate $z_2 - z_2' = 0$.
This results in expressions such as $[z_1, e_i, e_j] = q$.

If $q = 0$ we know that z_1 is in the $\mathbb{H}$ generated by e_i and e_j.
The intersection of two distinct $\mathbb{H}$ must be a copy of $\mathbb{C}$, and
The intersection of two distinct $\mathbb{C}$ must be a copy of $\mathbb{R}$.

If $q \neq 0$ we can still halve the space allowed for z_1.
This involves an explicit calculation.

Once we restrict z_1 to the correct dimension,
alternativity shows the resulting spaces are in the stated form.

Proposition 2.4.9 generalizes this to general x,
except for verifying the stated form of A when $d = 2$.

Results for $\mathbb{O}^n$

Proposition 2.3.4 is not the only difficulty we encounter.

Corollary (2.4.11, P.)

If S and T are closed subspaces, $S + T$ may not be.

Hilbert Spaces

A Hermitian space is **positive definite** if for all $x \in X$, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

This requires the image of $\langle x, x \rangle$ to be an ordered ring. Hence the characteristic of R must be 0.

Definition (2.5.1)

An **inner product space** is an alternative Hermitian space whose Hermitian form is positive definite.

Definition (2.6.1)

A **Hilbert space** is an inner product space that is complete relative to the topology induced by the order of R.

Orthogonal Projection

If S is a closed subspace we can define the **projection onto S**.

Theorem (2.6.2)

If X is a Hilbert space and S is a closed subspace of X, then for any $x \in X$ there is a unique $s = \text{proj}_S(x)$ in s minimizing $\langle x - s, x - s \rangle$.

Let S be the cycle generated by x.
Let y be ab arbitrary element of X.

Consider $z = \frac{\langle x, y \rangle}{\langle x, x \rangle} x$. In general $\langle x, y \rangle$ is an element of A.
Theorem 2.4.10 tells us that in general z is not in $x^{\perp\perp}$, however we still have $z \in \mathrm{Span}_A(S)$.

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Separability

A topological space, X, is called **separable** if it has a dense countable subset.

In the classical setting this is equivalent to having a countable basis, showing $X \cong l^2$.

Proposition (2.6.10, Ludkowski)

The octonionic l^2 space is separable.

Consequences

Theorem (2.6.5, Ludkowski)

A subspace S of a Hilbert space X is topologically closed iff $S = S^{\perp\perp}$.

It is not true that every octonionic Hilbert space has an $\overline{\varphi_{\mathbb{O}}}$-basis. However,

Theorem (2.6.11, P.)

Every octonionic Hilbert Space has an orthogonal basis.

If X is separable, this basis must be countable.

Unfortunately this does not guarantee that $X \cong l^2$ because the subspaces can be any form in Theorem 2.4.10.

Solèr's Theorem

Theorem (2.7.1, Solèr)

If X is an orthomodular Hermitian space over a skew-field K having an infinite orthonormal sequence, then K is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and X is a Hilbert space.

This excludes the octonions, as they are not a skew-field. Fortunately,

Theorem (2.7.9, P.)

If X is an orthomodular Hermitian space over a division pre-algebra K having an infinite orthonormal sequence, then K is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ and X is a Hilbert space.

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Theorem (3.3.1, Craig Nolder and P.)

The construction of Hopf fibrations over the division pre-algebras generalizes to the composition pre-algebras, yielding the fibrations below.

	Fibration	Alias
$\mathbb{R}$	$S^0 \to S^1 \to S^1$	
$\mathbb{C}$	$S^1 \to S^3 \to S^2$	
$\mathbb{C}^-$	$H^{0,1} \to H^{1,2} \to H^{1,1}$	$S^0 \times \mathbb{R} \to S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}$
$\mathbb{H}$	$S^3 \to S^7 \to S^4$	
$\mathbb{H}^-$	$H^{1,2} \to H^{3,4} \to H^{2,2}$	$S^1 \times \mathbb{R}^2 \to S^3 \times \mathbb{R}^4 \to S^2 \times \mathbb{R}^2$
$\mathbb{O}$	$S^7 \to S^{15} \to S^8$	
$\mathbb{O}^-$	$H^{3,4} \to H^{7,8} \to H^{4,4}$	$S^3 \times \mathbb{R}^4 \to S^7 \times \mathbb{R}^8 \to S^4 \times \mathbb{R}^4$

Table: Hopf fibrations for all composition pre-algebras over $\mathbb{R}$.

Split Octonionic Cauchy Integral Formula

Theorem (4.4.5, P.)

Let Ω be an open connected region in $\mathbb{O}^-$. Let $M \subset \Omega$ be compact with smooth boundary ∂M that intersects $N(q) = 0$ transversely. Let $f \in C^5(\Omega, \mathbb{O}^-)$ be left (right) $\mathbb{O}^-$-regular on Ω. Define $\Phi_\varepsilon(q) = \dfrac{3}{\pi^4} \dfrac{\overline{q}^\dagger + i\varepsilon\overline{q}}{(N(q) + i\varepsilon\|q\|^2)^4}$. *Then:*

$$\lim_{\varepsilon \to 0} \int_{\partial M} \Phi_\varepsilon(q - q_0)(\star(dq) f(q)) = 1_M(q_0) f(q_0),$$

$$\left(\lim_{\varepsilon \to 0} \int_{\partial M} (f(q) \star (dq)) \Phi_\varepsilon(q - q_0) = 1_M(q_0) f(q_0) \right).$$

Grave's Question

Hamilton told his friend Graves about his discovery of the quaternions, and Graves responded "If with your alchemy we can produce three pounds of gold, why stop there?"

He then produced the octonions, anticipating an infinite family of composition algebras.

"If with your alchemy we can produce seven pounds of gold, why stop there?"

There Be Dragons!

After the octonions, the Cayley-Dickson construction breaks alternativity and the composition property.
These two properties are significant for a majority of the octonionic results above.

Any attempt to extend the octonions through a bilinear Cayley-Dickson like doubling process will contain the pseudo-octonions.
This introduces zero divisors with multiplicative inverses.

Johnathan Smith produced an alternative product that does generate an infinite family of composition algebras.
This is neither bilinear nor continuous.

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"Nobody has managed to develop a good theory of octonionic linear algerba." – Baez

This is based on two gaps, the lack of a definition of modules and linear maps between modules.

The pre-modules defined here allow for definitions of octonionic projections, a proof that every octonionic Hilbert space has a basis and a generalization of Soler's theorem.

The elementary row operations allow for row reduction and a classification of the cyclic spaces.

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End

Thank you!